ENSHMG February 6-9, 2006
an EU SOCRATES short course
on

Engineering Continuum Mechanics (1) Traffic flow theory

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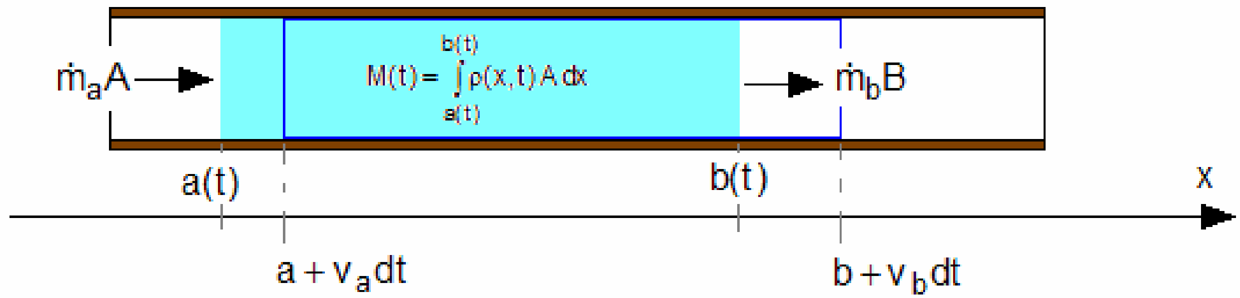
EU RTN DIGA

Mass balance: Traffic flow theory

Traffic flow theory:

R. Haberman, Elementary Applied Partial Differential Equations, Prentice-Hall,1998

Mass balance



$$M(t)=\int_{a(t)}^{b(t)}\rho(x,t)\,A\,dx \qquad \dot{M}=\frac{dM}{dt}$$

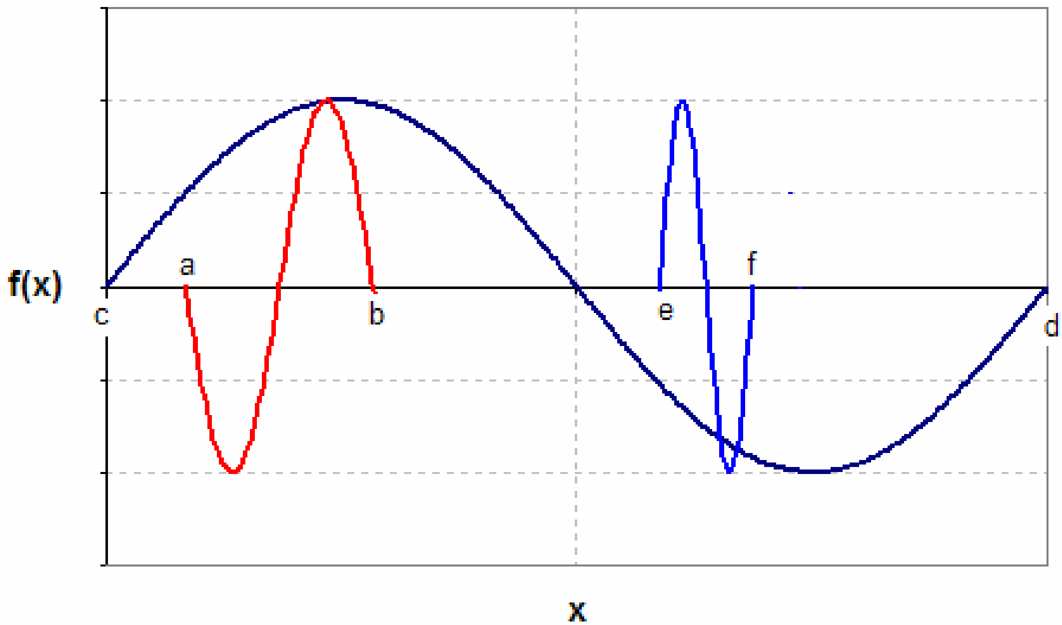
$$\frac{\dot{M}}{A}=\frac{d}{dt}\int_{a(t)}^{b(t)}\rho(x,t)dx=\int_{a(t)}^{b(t)}\frac{\partial}{\partial t}\rho(x,t)dx+\frac{db}{dt}\rho(b,t)-\frac{da}{dt}\rho(a,t) \qquad \frac{da}{dt}=v(a,t)\ ,\ \frac{db}{dt}=v(b,t)$$

$$\frac{\dot{M}}{A}=\int_{a(t)}^{b(t)}\frac{\partial}{\partial t}\rho(x,t)dx+\left[\rho v\right]_{a(t)}^{b(t)}=\int_{a(t)}^{b(t)}\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)\right]dx$$

a fundamental theorem of analysis

$$f(x) \text{ continuous} \quad \forall x \in [c,d]$$

$$\int_a^b f(x)\,dx = 0 \quad \forall a,b \quad c<a<b<d \quad \Rightarrow \quad f(x)=0 \quad \forall x \in [c,d]$$



Conservation of mass

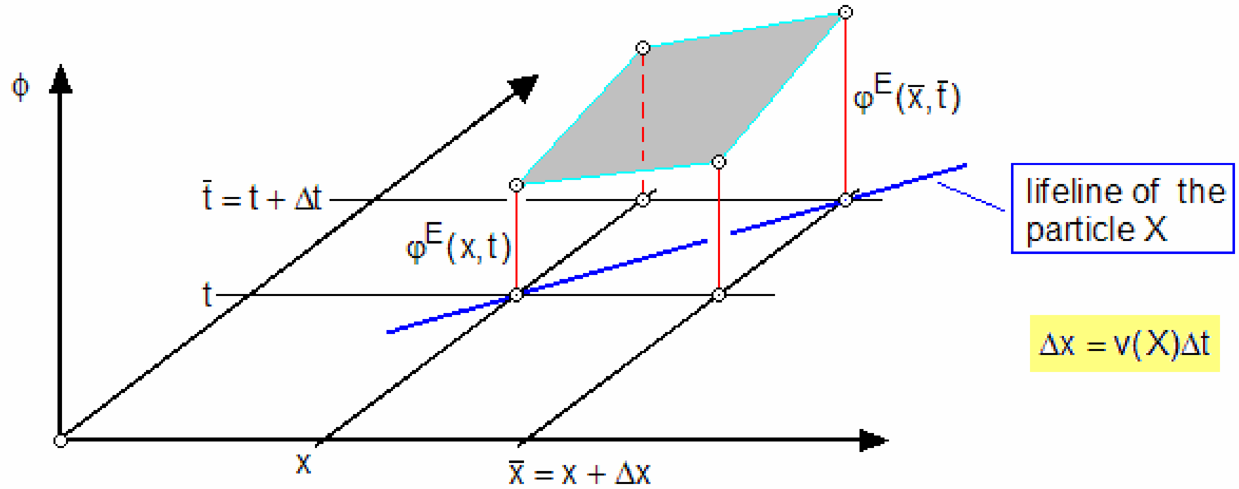
$$\dot{M} = \frac{dM}{dt} = 0$$

$$\int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \right] dx = 0$$

$$\int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \right] dx = 0 \quad \forall [a,b] \quad \Leftrightarrow$$

$$\underline{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0} \quad \forall x \in [a,b]$$

the material time derivative



$$\phi(x + v\Delta t, t + \Delta t) \approx \phi(x,t) + \left(\frac{\partial \phi}{\partial x}\right) v\Delta t + \left(\frac{\partial \phi}{\partial t}\right)\Delta t$$

$$\frac{D\phi}{Dt} = \lim_{\Delta t \to 0}\left(\frac{\phi(x + v\Delta t, t + \Delta t) - \phi(x,t)}{\Delta t}\right) = \left(\frac{\partial \phi}{\partial x}\right) v + \left(\frac{\partial \phi}{\partial t}\right)$$

mass balance equation

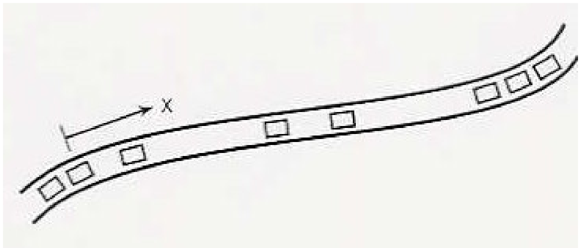
$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{D\rho}{Dt} + \rho \,\mathrm{div}\,\vec{v} = 0$$

The elementary traffic-flow theory

M.J. Lighthill and G.B. Whitham(1955).



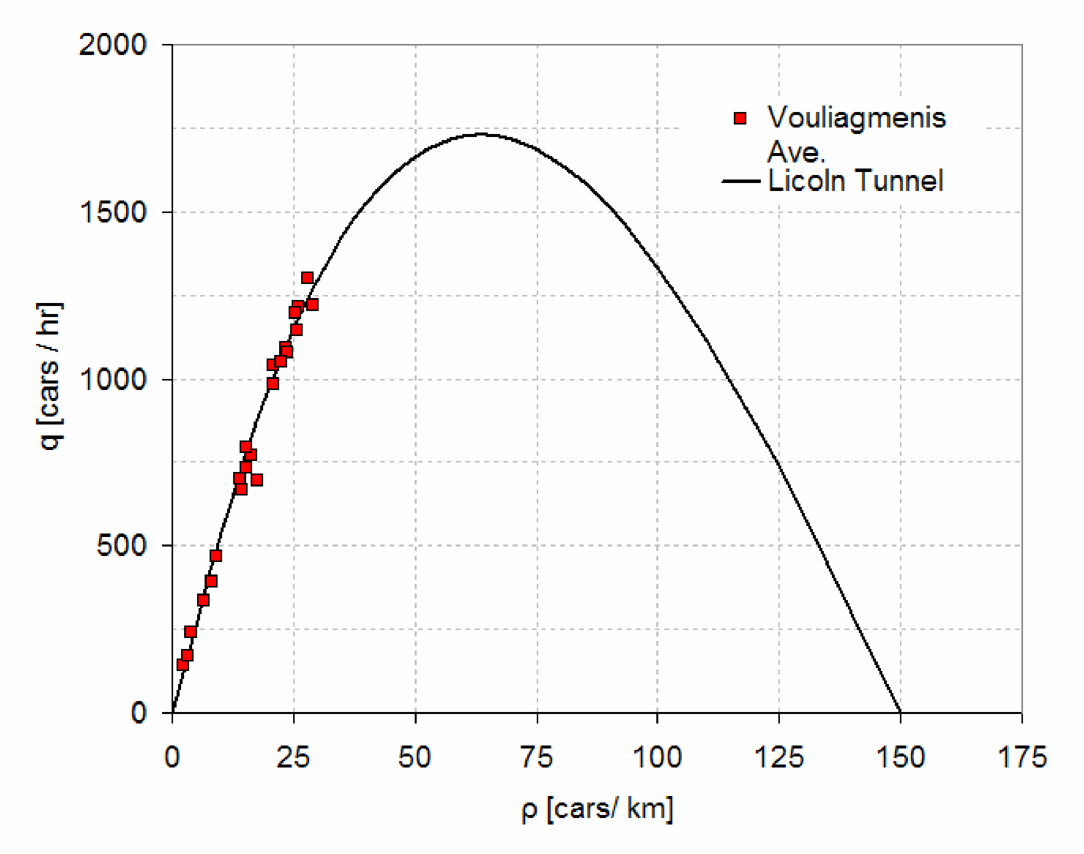
- Linear density of vehicles: $\rho = \rho(x, t)$, $[\rho] = \text{cars} / \text{km}$

- (mean) speed of cars: $v = v(x,t)$, $[v] = \text{km} / \text{hr}$

- "mass" balance: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$

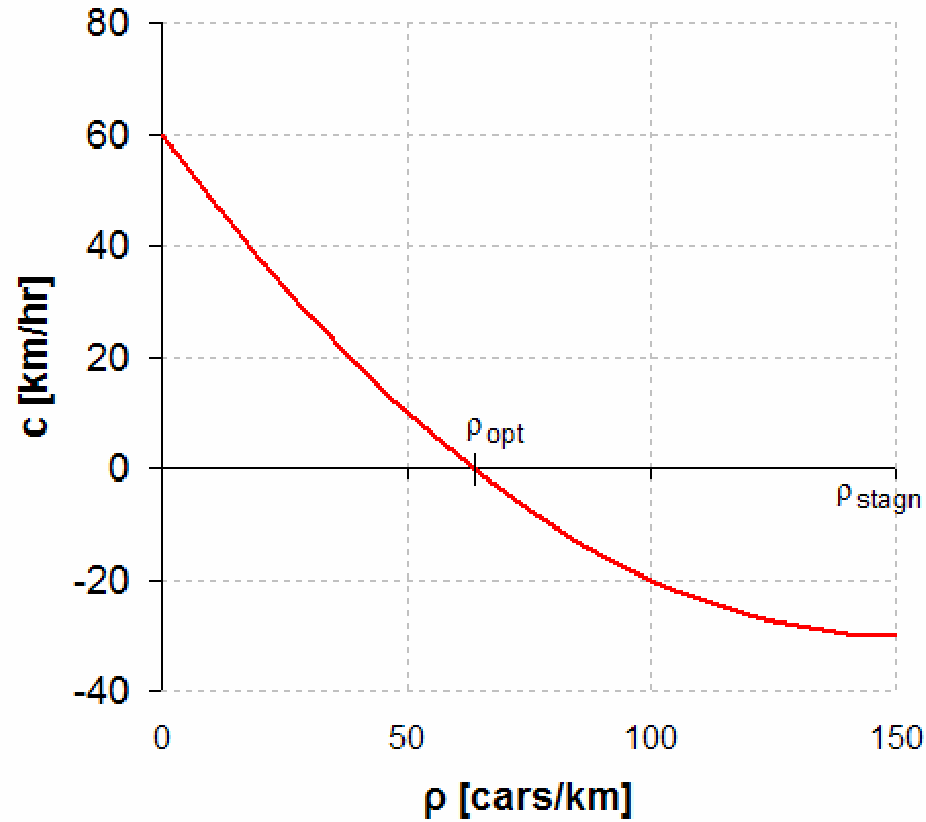
- need for a "closure"; eg. for the car flow-rate: $q = \rho v = Q(\rho, \ldots)$

empirical constitutive law



the celerity

- $v = V(\rho)$
- $\boxed{q = Q(\rho) = \rho\, V(\rho)}$; e.g. $q = Q(\rho) = 60\,\rho - \frac{3}{5}\rho^2 + \frac{1}{750}\rho^3$
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{dQ}{d\rho}\frac{\partial \rho}{\partial x} = 0$
- $\boxed{\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0}$; $c = C(\rho) = \frac{dQ}{d\rho}$



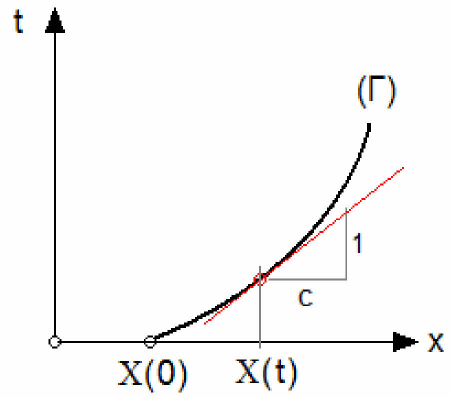
the traffic flow problem

Given: $\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$ (*)

- Assume that at time $t = 0$ the *initial condition*: $\rho(x,0) = r(x)$.
- Compute the space-time evolution of $\rho(x,t)$

Solution: Consider in the **plane of events** $O(x,t)$ a curve

$$(\Gamma): \quad x = X(t) \quad \text{s.t.} \quad \frac{dX}{dt} = c\left(H\left(X(t),t\right)\right)$$



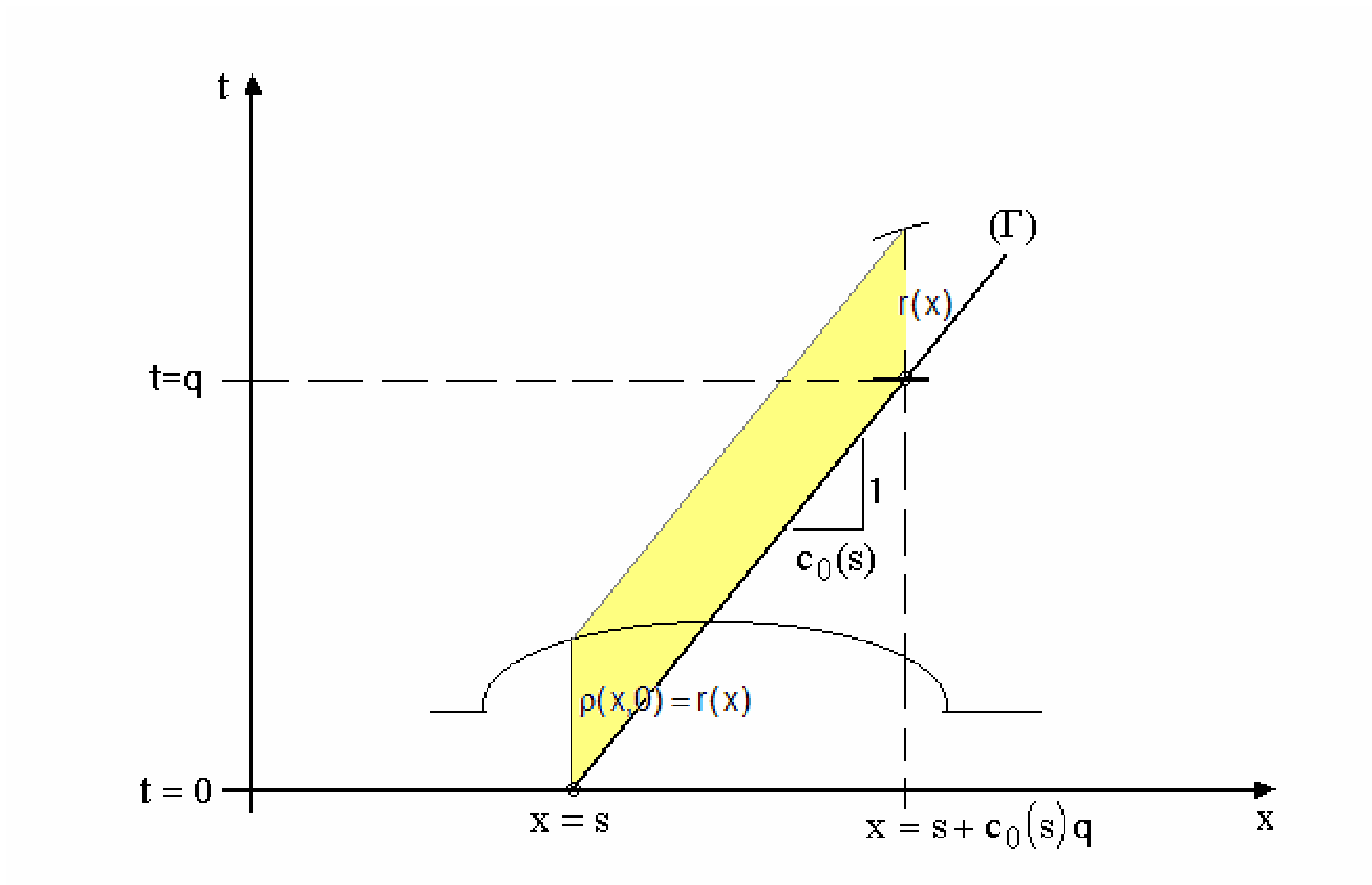
Along (Γ) the density is a function only of time t: $\rho = \rho\left(X(t), t\right) = \hat{\rho}(t)$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dX}{dt} = 0 \quad \Rightarrow \quad \rho = \text{const}$$

$$\Rightarrow c(\hat{H}) = \text{const.} \quad \Rightarrow \quad \frac{dX}{dt} = \text{const.}$$

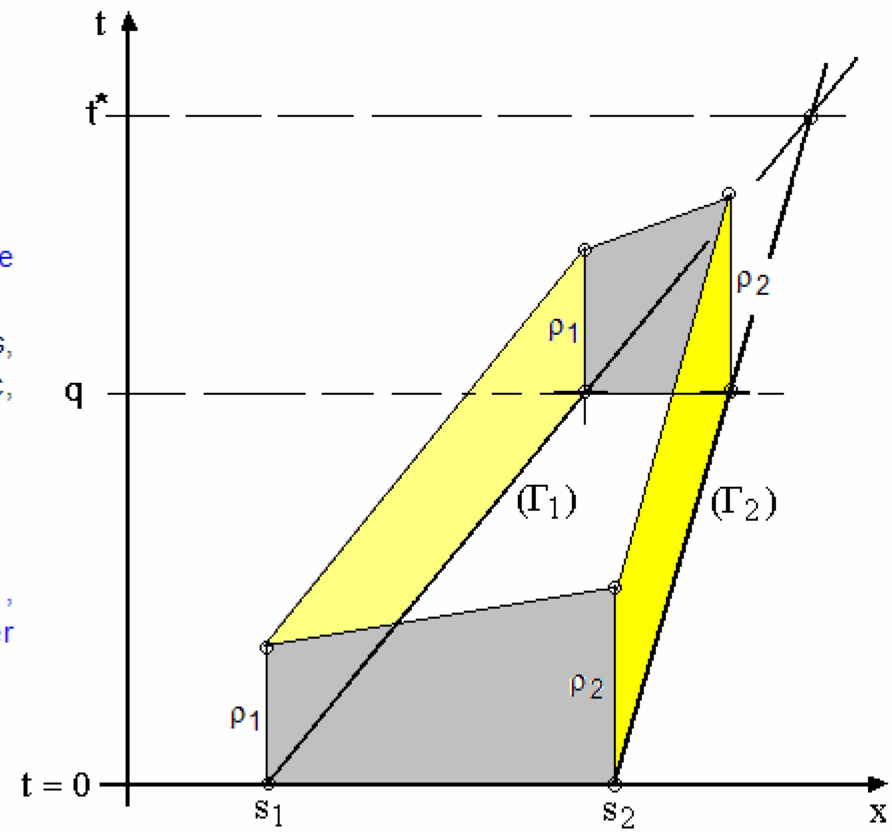
(Γ) is a **straight line** in the plane $O(x,t)$ of events.

(Γ) is called a **characteristic line** or simply a **characteristic** of equation (*)[1].

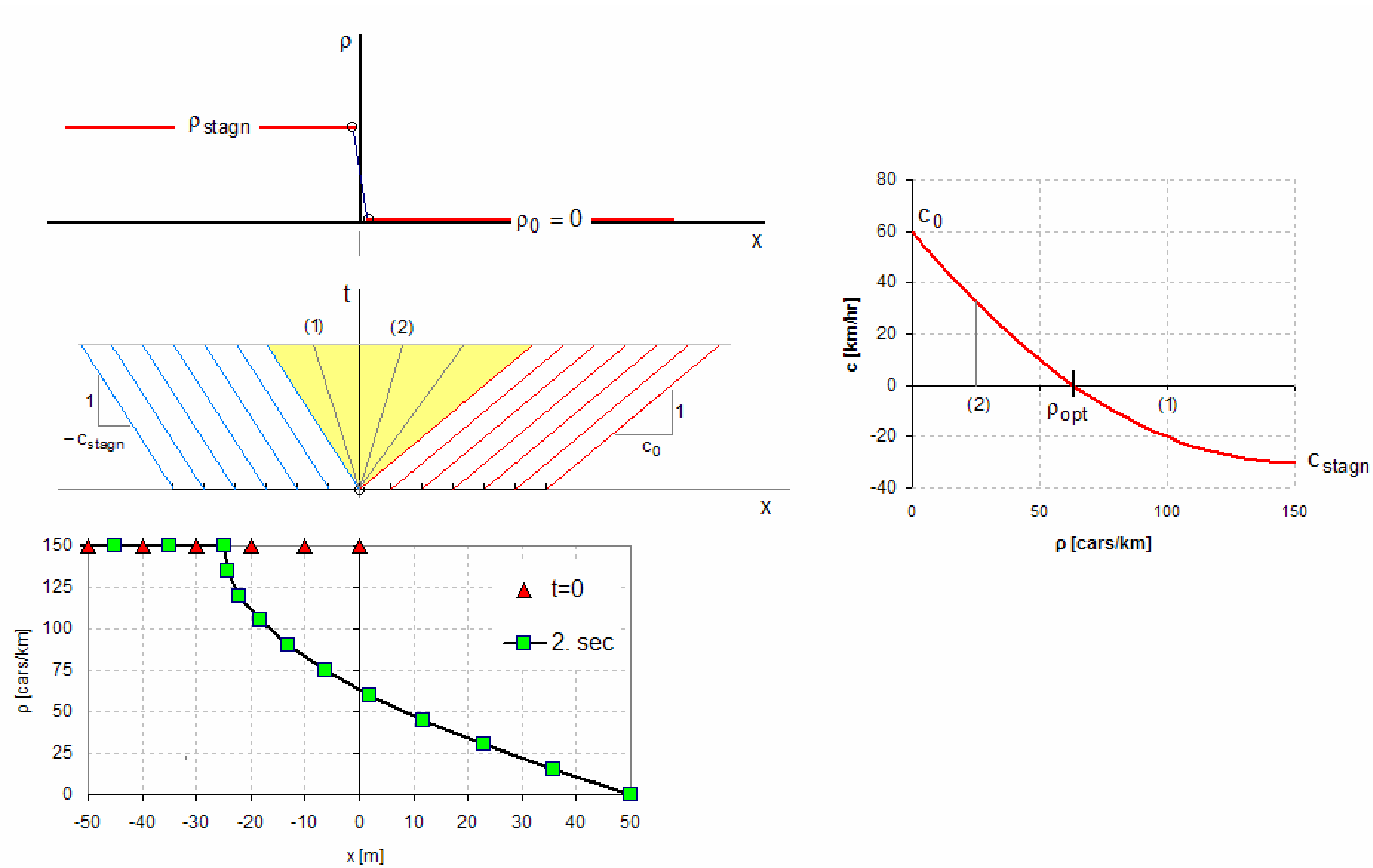

t
t=q
t = 0
(Γ)
r(x)
1
c0(s)
ρ(x,0) = r(x)
x = s
x = s + c0(s)q
x

'Method of Characteristics':

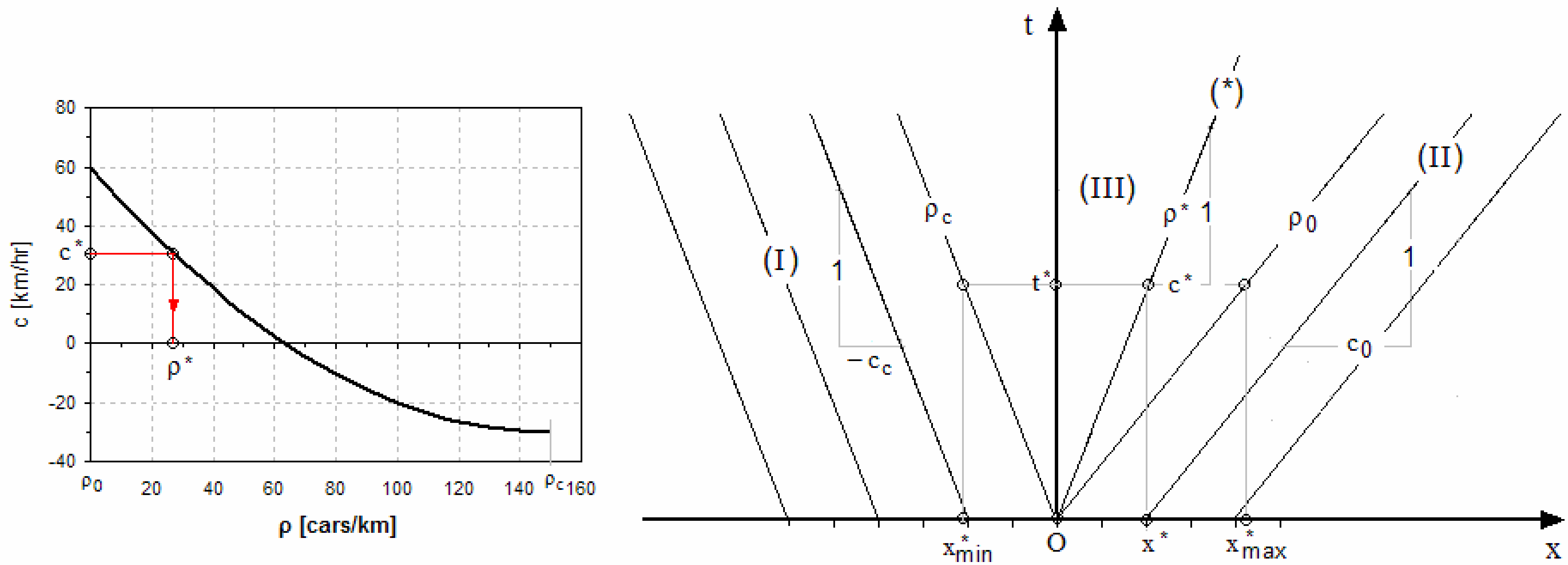
1. The x-axis is discretised.
2. We evaluate the initial condition for the height at the discrete points on the x-axis, $\rho(x,0) = r(x)$
3. At any individual point $x = s$, along the x-axis ($t = 0$), we compute the propagation velocity, $c_0(s) = c(r(s))$
4. In the plane of events $O(x,t)$ and through the point on the x-axis, with co-ordinates $(s,0)$ we draw the straight-line characteristic, $(\Gamma): \quad x = s + c_0(s)t$
5. Along this characteristic straight line the information concerning the density is transferred constant, $(\Gamma): \quad \rho(s + c_0(s)t, t) = \rho(s,0) = r(s)$
6. If we want to evaluate the density profile at a given time $t = q > 0$, we intersect the characteristics with the line, $t = q$, and plot over the intersection points the values for ρ which they carry.



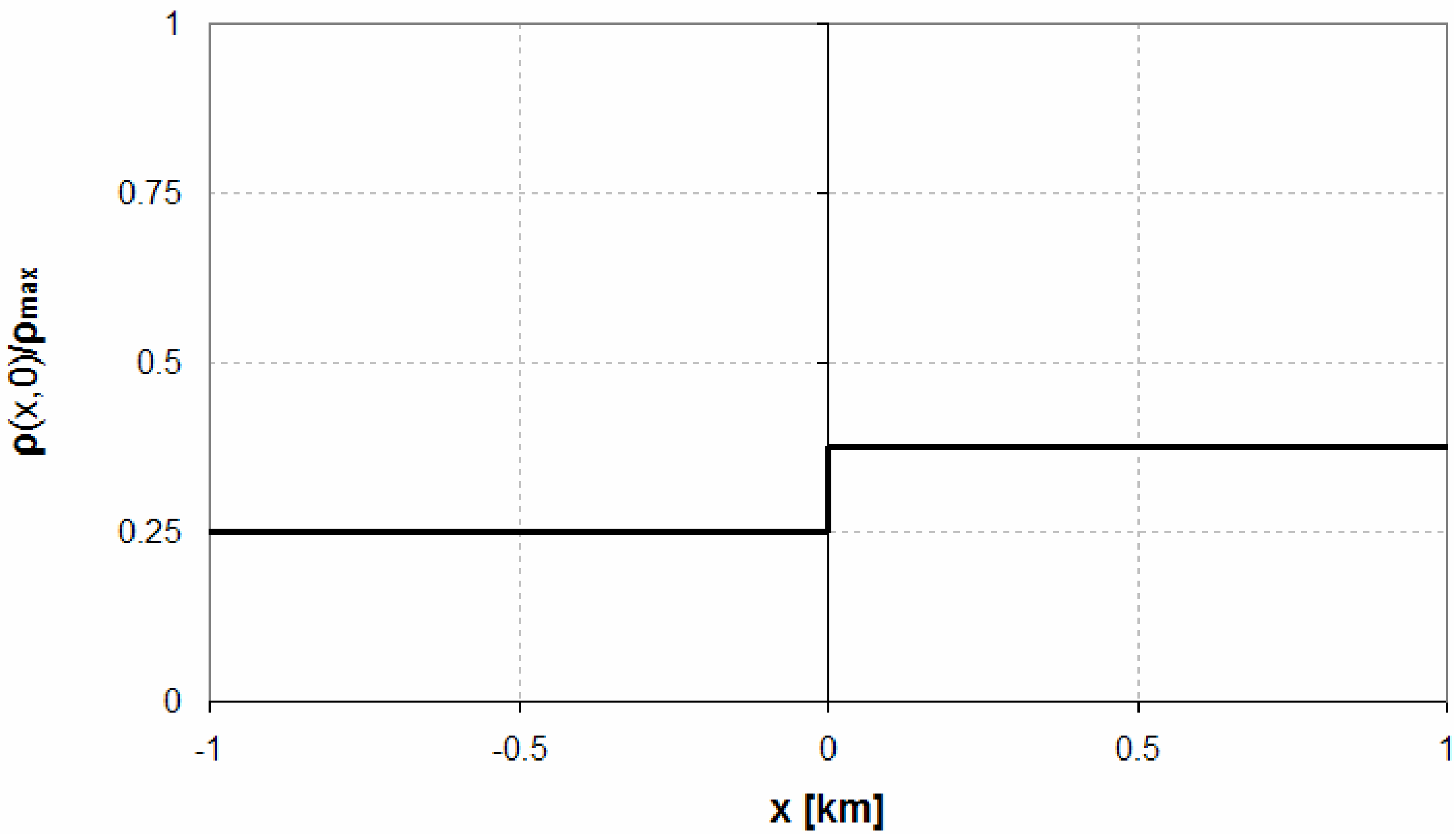
The traffic light problem

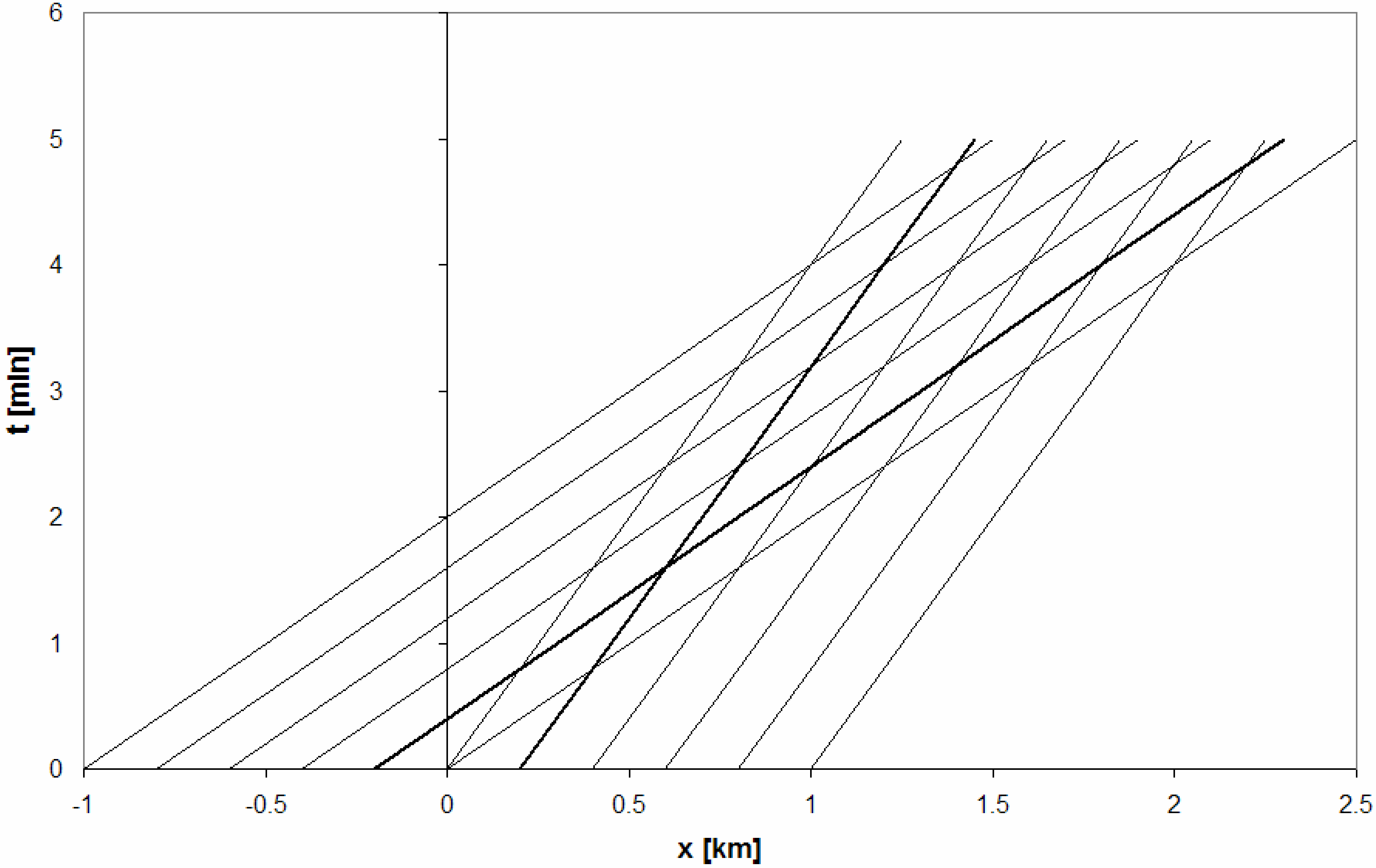


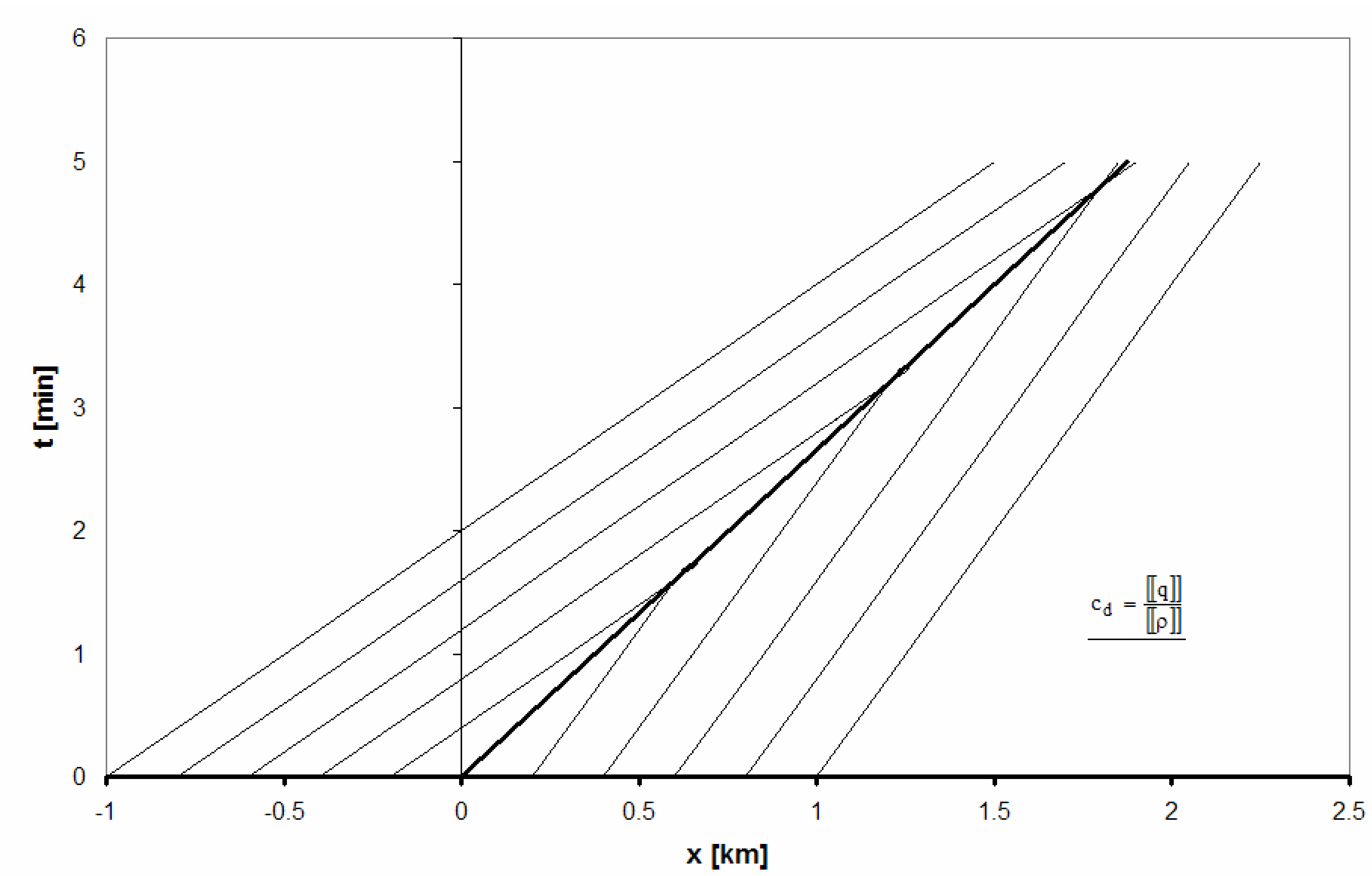
the expansion fan



the shock wave




6
5
4
3
2
1
0
t [min]
-1
-0.5
0
0.5
1
1.5
2
2.5
x [km]



6
5
4
3
2
1
0
t [min]
-1
-0.5
0
0.5
1
1.5
2
2.5
x [km]
$c_d = \frac{[[q]]}{[[\rho]]}$

1st Modification: “Viscosity” correction (the driver looks ahead; stabilizing)

2nd Modification: Reaction time (destabilizing)

Viscosity correction

If the flow ahead is getting denser or looser, the driver is able to adjust the speed of the vehicle accordingly:

$$v = V(\rho) - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x}$$

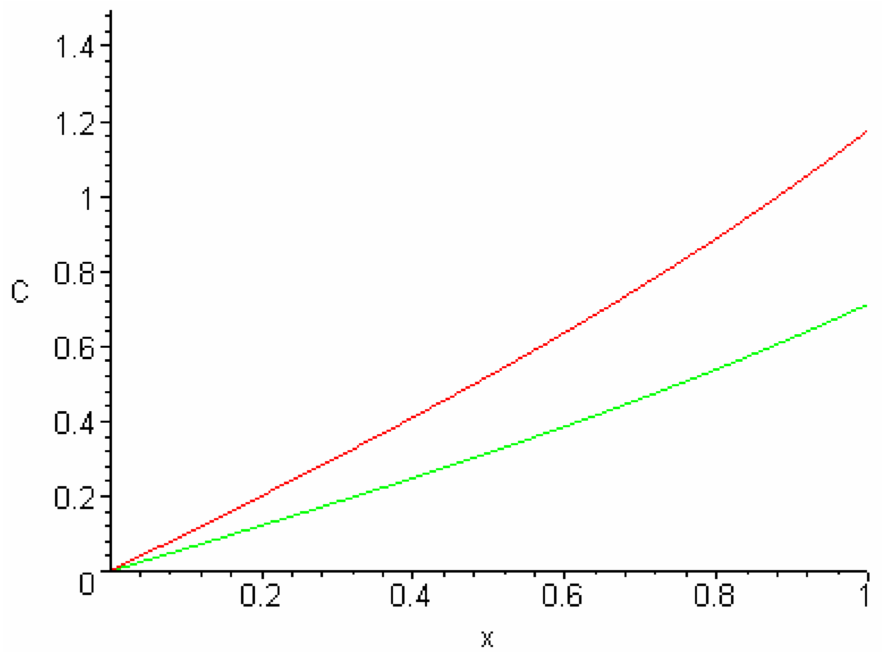
$$\nu = \ell_c v_0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \quad \text{(BURGER Eq.)}$$

Canonical form of the Burger equation

$$C = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) \quad \Rightarrow$$

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = \nu \frac{\partial^2 C}{\partial x^2}$$



Reaction time

$$v = V\big(\rho(x, t-\tau)\big) \approx V(\rho) - \tau V'(\rho)\frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t} + C(\rho)\frac{\partial\rho}{\partial x} = \tau\frac{\partial}{\partial x}\left(\rho V'(\rho)\frac{\partial\rho}{\partial t}\right)$$

$$\rho = \rho^{*} + \tilde{\rho}(x,t)$$

$$\frac{\partial\tilde{\rho}}{\partial t} + c^{*}\frac{\partial\tilde{\rho}}{\partial x} = \tau\rho^{*}V'(\rho^{*})\frac{\partial^{2}\tilde{\rho}}{\partial x\partial t}$$

Linear stability analysis

$$\tilde{\rho} = \mathrm{Re}\big(\exp(ikx + st)\big) \quad , \quad i = \sqrt{-1}$$

$$s = \frac{\tau k^2 \rho^* c V'(\rho^*) - ikc^*}{1 + \left(\tau k \rho^* V'(\rho^*)\right)^2}$$

with V'<0 instability is predicted at heavy traffic conditions (c<0)