ENSHMG

Short Course on

ENGINEERING CONTINUUM MECHANICS

with applications from
Fluid- and Soil Mechanics

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Table of Contents

I. Preface

This book is designed as an introductory course in mathematical modeling in engineering continuum mechanics. The book stems from the lecture notes of a course, which is presently taught by the author to 5th semester undergraduate students of Civil Engineering and Applied Mathematics and Physics at the National Technical University of Athens. The contents of the book are built on the three main principles of Mechanics[1]: Mass Conservation, Conservation of Linear Momentum and Conservation of Energy. Mainly one-dimensional problems are treated here, which lead to the formulation and solution of elementary partial differential equations, with respect to one spatial coordinate and to time.

The principle of mass balance will be used for the formulation of the continuity and storage equations for hydro- and soil-mechanics problems. The conservation of (linear) momentum principle will be explained using examples from shallow water- wave theory. Both principles will be used to introduce Darcian flow and pore-fluid pressure diffusion in porous media. In the last section the energy balance law will be used for the formulation of the elementary theory of heat transfer in continuous media. As an application of the theory a thermo-mechanical instability in steady shearing of a dissipative medium will be presented.

The book contains solved problems and a number of selected exercises that complement the theory and help the student to become familiar with the various continuum-mechanical concepts.

Ioannis Vardoulakis,
Paiania, January 2001

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[1] F. Ziegler, *Didaktische Aspekte in mechanischen Erhaltungssätzen*, GAMM-Mitteilungen, Heft 1,61-72, 1998.

II. Selected Books of Reference

1. Haberman, R., *Elementary Applied Partial Differential Equations*, Prentice-Hall, sect. 12.6.2, 1998.

This is an excellent textbook on **partial differential equations**, with many solved problems and useful solution techniques. It is written in a way that appeals to the Engineering and Applied Mathematics student.

2. Roberts, A.J., A One-dimensional Introduction to Continuum Mechanics, World Scientific, 1994.

This book has motivated the author to introduce his new **One-dimensional Continuum Mechanics** course for undergraduate Engineering students at N.T.U.A. Robert's book covers a wider range of applications, and it appears to be designed mostly for Applied Mathematicians.

3. Segel, L.A., *Mathematics Applied to Continuum Mechanics,* Dover 1977.

We refer to this excellent textbook, since one can find there - among other things - a very clear and rigorous presentation of the **water-waves theory**. It will be useful to read the chapters that deal with dimensional analysis and linearization.

4. Sommerfeld, A., *Mechanik der deformierbaren Medien,* 1944, Verlag Harri Deutsch (1992).

Sommerfeld's "Mechanik", reprinted from its initial 1948 edition, is a "classic" reading and gives an insight as of how the great Physicist taught deformable bodies mechanics and especially **Fluid Mechanics**.

5. Taylor, D.W., *Fundamentals of Soil Mechanics*, Wiley, 1948.

The **Soil Mechanics** of Taylor may be out of print, but has not found yet its replacement. It's a "must" not only for the Civil Engineer but also for anyone, who wants to grasp the essence of Soil Mechanics.

6. Whitham, G.B., *Linear and Non-Linear Waves,* John Wiley & Sons, 1974.

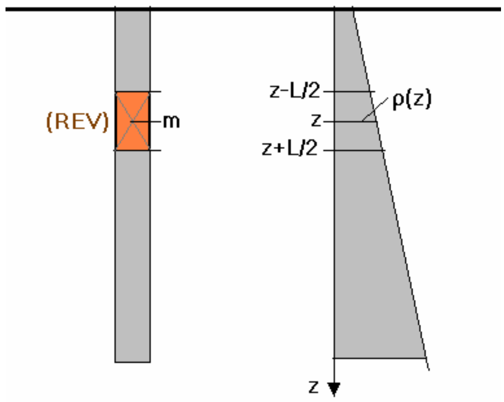
In this valuable textbook the mathematical theory of **non-linear water waves** is given. The reader must do a lot of side-work in order to appreciate in full the richness of results contained therein.

THE CONTINUUM

1.1 The Density

We consider a material body whose mass density ρ we want to determine. For example we want to determine the density distribution (or in technical terms, the unit weight distribution) of a natural soil body that may vary with depth z. For the determination of the density at any given depth we choose first a *'sampling length'* L around the sampling point at depth z. Secondly we measure the mass $m(z,L)$ of the *sample* (with length L and cross-section A) and finally we compute the mass density from the formula

$$< \rho > = \frac{m(z,L)}{LA} \tag{1.1}$$



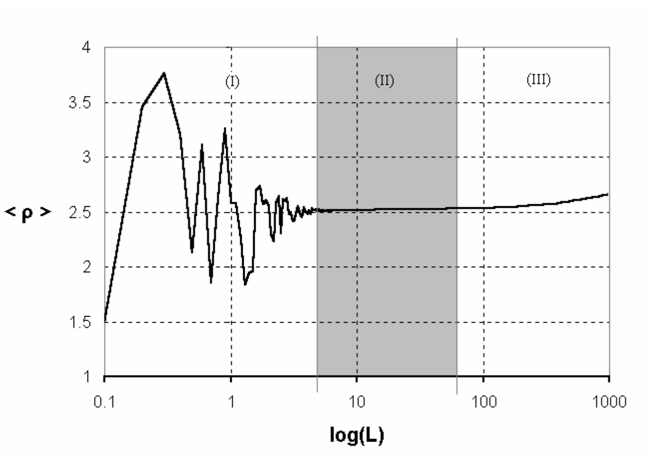
In normally consolidated clay deposits the density is usually varying linearly with depth

As can be seen from the sketch below, the average density, as computed from eq. (1), will be a function not only of position but also of the sampling length,

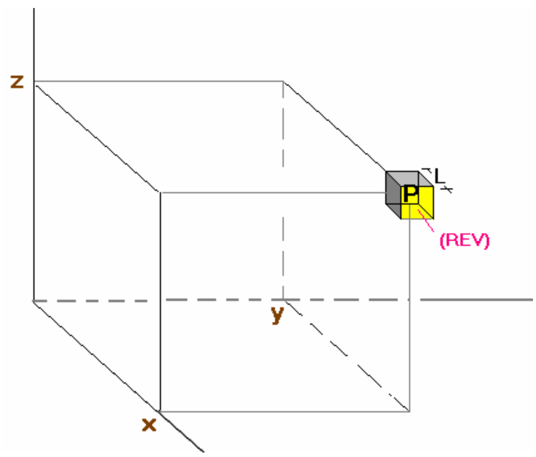
$$<\rho> = \tilde{\rho}(z,L)$$

In general we will find that for 'small' sampling windows one finds itself in region (I) of granular (molecular) fluctuations, whereas for very large L we are in region (III) of macroscopic variations. There is however an intermediate region (II) where the average density is not affected by the sampling length. For example, if the density variation is indeed linear with z, then from the trapezoidal rule it follows immediately that the mean value, independently of the sampling length, coincides with the local value at the center of the sampling length,

$$<\rho>_{z,L} = \rho(z) \qquad (1.2)$$



Connection between mean density and the volume from which it is derived[1]



[1] cf. L. Prandtl and O.G. Tietjens, *Fundamentals of Hydro- and Aerodynamics*, Dover , 1934.

According to Euler's original proposition[2] we introduce in Continuum Mechanics the concept of *material point.* Geometrically the material point is mapped in $\Re^3$ on to the geometric center ? (x,y,z) of an elementary cube with dimension L. This elementary material cube is called *'representative elementary volume'* (REV). According to the remarks made above, we assume that this cube is sufficiently large so that granular fluctuations are smoothened out and sufficiently small so that macroscopic changes are not affecting the result.

For example, in a soil the minimum dimensions of the REV are set by the grain size D_g. It is found that an assembly of $10^2 \div 10^3$ sand grains possess enough variability, so that averages are representative. On the other hand samples taken from a site should not be larger that the natural soil layer deposition thickness, mostly determined by topography, climatic history and rock mineralogy. This upper bound is usually in the decimeter range.

Following this general procedure, the average value of a quantity like the density over the REV is mapped, according to eq. (1.2), onto the 'material' point that occupies the center of the corresponding REV,

$$P(x,y,z) \to \rho = \langle \rho \rangle_{REV}$$

Within the frame of Continuum Mechanics we will assume that such a density distribution function $\rho = \rho(x,y,z)$ is at least piecewise continuous and that discontinuities or *'shocks'* may occurr along discrete surfaces. The density of a continuous medium is in general neither constant in space nor in time. In order to demonstrate this fact we study in the next section the density variation inside a soil-water mixture during sedimentation.

1.2 Density Variation of a Soil-Water Mixture during Sedimentation

We consider a column of water within which we have mixed a small quantity of a granular, non-dissolving material, like e.g. a small amount of clay or fine silt. After rigorous mixing, we let this homogeneous water-clay mixture at time $t = 0$ at rest and we observe changes in coloration in the mixture. The upper water layers become lighter in color, indicating that sedimentation of the soil particles due to gravity is under way.

Within the realm of Continuum Mechanics, a *mixture* consisting of water and soil grains may be considered as a single continuum[3], consisting of *two*

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[2] Leonhard Euler 1707-1783.

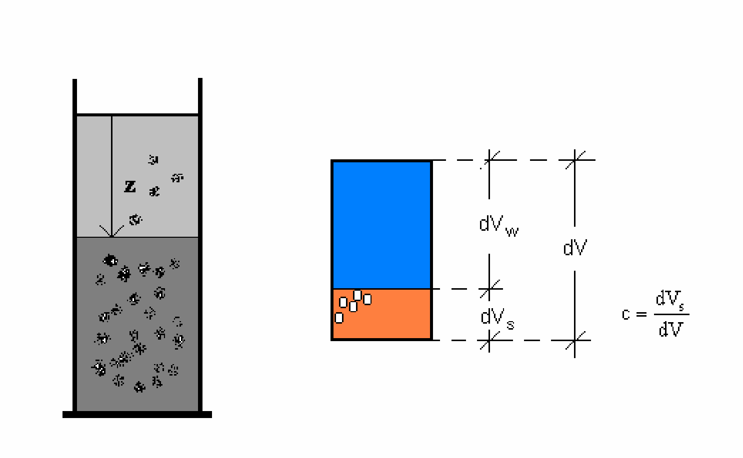
[3] In the Mechanics of Materials we distinguish among *continuum* and *discrete models*. Such a distinction is of course an ancient one, which can be traced back to the philosophical controversy between atomists and stoics (see e.g. Lloyd, G.E.R.: *Early Greek Science*, Norton, 1970). In our times this controversy still persists among those who believe that quantities, which enter a continuum description should be seen as ‘averages’ of some other underlying ‘micro-scopic’ properties of the material, and those who do not accept this point of view.

phases. As it is represented in the corresponding *volume fraction diagram*, each phase occupies at any point and time a particular percentage of the total representative volume dV of the mixture, i.e.

$$dV = dV_W + dV_S$$

where the

- 1^{st} phase - water is occupying the volume dV_W (w *water*)
- 2^{nd} phase - solid grains is occupying the volume dV_S (s *solid*)



We define further

1. The volume concentration of solids

$$c = \frac{dV_S}{dV}$$

2. The density of the mixture

$$\bar{\rho} = \frac{dm_W + dm_S}{dV}$$

where dm_W and dm_S are the masses of the two corresponding phases in the representative volume dV.

The densities of the two constituents are

$$\rho_S = \frac{dm_S}{dV_S} \quad , \quad \rho_W = \frac{dm_W}{dV_W}$$

Thus the mixture's density becomes,

$$\bar{\rho} = (1-c)\rho_w + c\rho_s \tag{1.3}$$

Above relationship means, that within the frame of Continuum Mechanics the water-soil mixture is a continuum that has the following fundamental property: At each point in space water and solids coexist at percentages $c_w = 1 - c$ and $c_s = c$, receptively. The densities ρ_s and ρ_w of the constituents are assumed to be constant, whereas the mixture's density is changing according to eq. (1.3), since the solids concentration is variable,

$$c = c(z,t)$$

For the computation of the field

$$\overline{\rho} = \overline{\rho}(z,t)$$

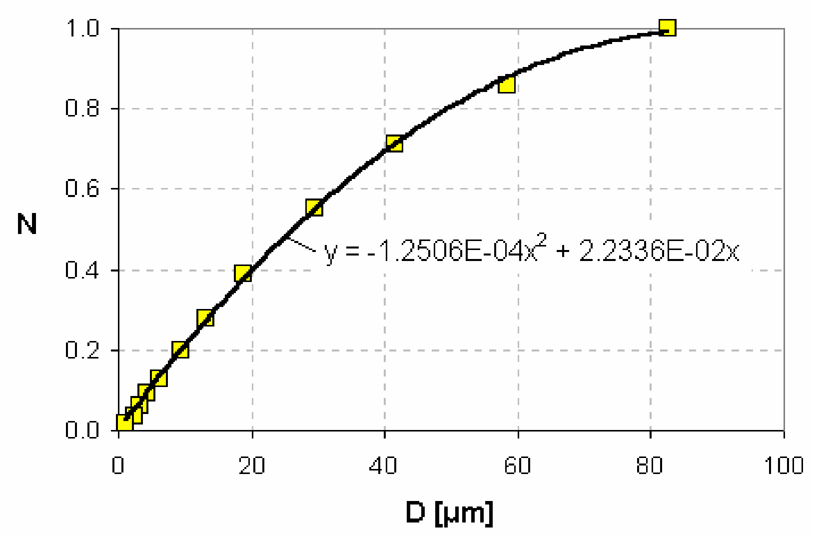
we consider a granular material, with given grain-size distribution curve. This is the cumulative distribution of the characteristic dimension (diameter) D of the various grains that constitute the soil. This distribution is described mathematically by the so-called *sieve-curve,* which is a cumulative distribution curve of the form

$$N = N(x < D)$$

? is the percentage in weight of the granular material (soil) that has grains with a diameter less than D. As an example we give below the sieve curve of a particular clay[4] which is approximated by the following empirical polynomial law,

$$N \approx a\left(\frac{D}{D_{ref}} + \left(\frac{D}{D_{ref}}\right)^2\right)$$

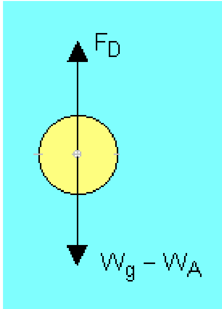
with : $a = 0.2223 \quad , \quad D_{ref} = 13.33\ \mu m$



[4] D.W. Taylor, *Fundamentals of Soil Mechanics*, Wiley, p.46, 1948

According to Stoke's law almost spherical particles settle down inside a fluid at rest with a terminal velocity v_{lim}, which depends on the diameter D of the particle and on the fluid viscosity as

$$v_{lim} = \frac{\rho_s/\rho_w - 1}{18\nu} g D^2 = \lambda D^2$$



- ρ_s: is the density of the particle (for quartz $\rho_s = 2.7\,gr/cm^3$)
- ρ_w: is the fluid density (for water $\rho_w = 1.0\,gr/cm^3$)
- ν: is the kinematic viscosity of the fluid (for water $\nu = 1.0\,mm^2/sec$)
- g: is the acceleration of gravity (g = 9.81 m/sec^2)

In the considered case we have,

$$\lambda = \frac{(2.7-1)\cdot 9.81\cdot 10^3}{18} \frac{1}{10^3\ \mu m\cdot sec} = 0.9265 \frac{1}{\mu m\cdot sec}$$

At time t, after the begin of the deposition process, and for a column reaching the depth z from the surface, there will be no particles with size grater than the characteristic size D_t, which corresponds to the smallest grain that may travel all the distance z during this time interval,

$$z = v_{lim} t = \lambda D_t^2 t \quad \Rightarrow \underline{D_t = \sqrt{\frac{1}{\lambda}\frac{z}{t}}} \qquad (1.4)$$

This procedure does hold neither for very small times nor for small distances from the free surface: $t_{min} > \lambda D_{max}$, $z_{min} > D_{max}$. This means that for times $t > t_{min}$ and depths $z > z_{min}$ the solids concentration is

$$c(z,t) = N(x < D_l)\, c_0$$

where c_0 is the initial concentration of solids in suspension.

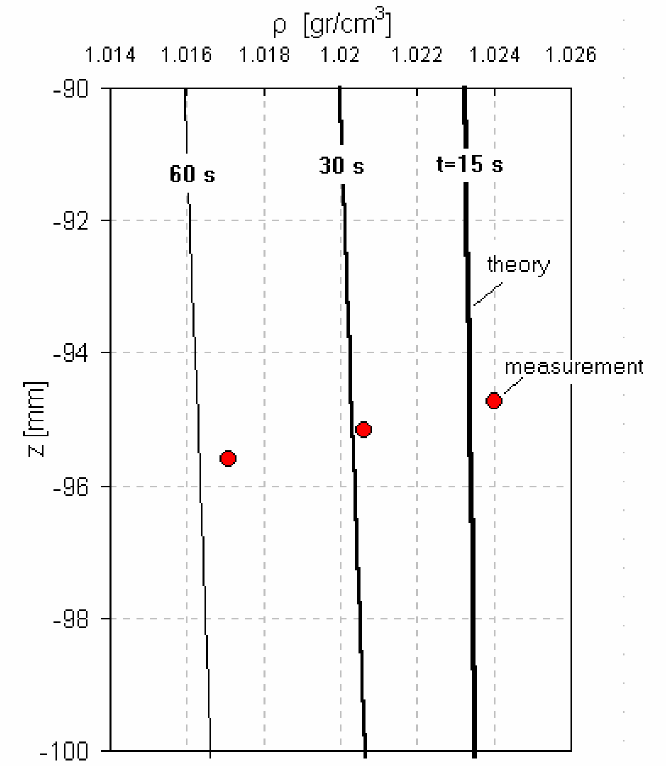
Above equations result finally into the following density function,

$$\bar{\rho} = \rho_w \left(1 + \left(\frac{\rho_s}{\rho_w} - 1 \right) c_0 N \right)$$

where in the considered example

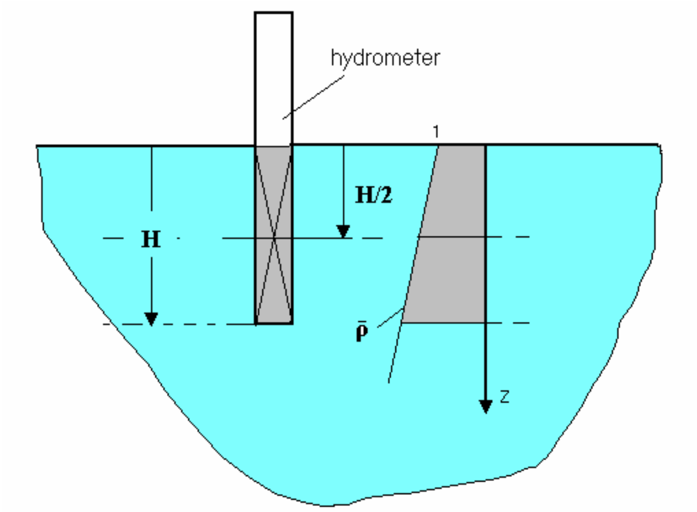
$$N \approx a \left(\frac{D_t}{D_{ref}} + \left(\frac{D_t}{D_{ref}} \right)^2 \right)$$

and D_t is given by eq. (1.4).



In the considered example the initial concentration of $c_0 = 0.01141$. As shown in the plot, in relatively short time the mixture's density distribution becomes approximately linear with depth. This observation has lead Casagrande (1931)[5] to suggest an inverse technique to determine the sieve-curve of a given soil from pycnometric measurements. Indeed the hydrometer method is based on the assumption that the average density for an immersion depth H equals to the density at mid-depth ? /2.

[5] D.W. Taylor, *Fundamentals of Soil Mechanics*, Wiley, p.36-ff, 1948



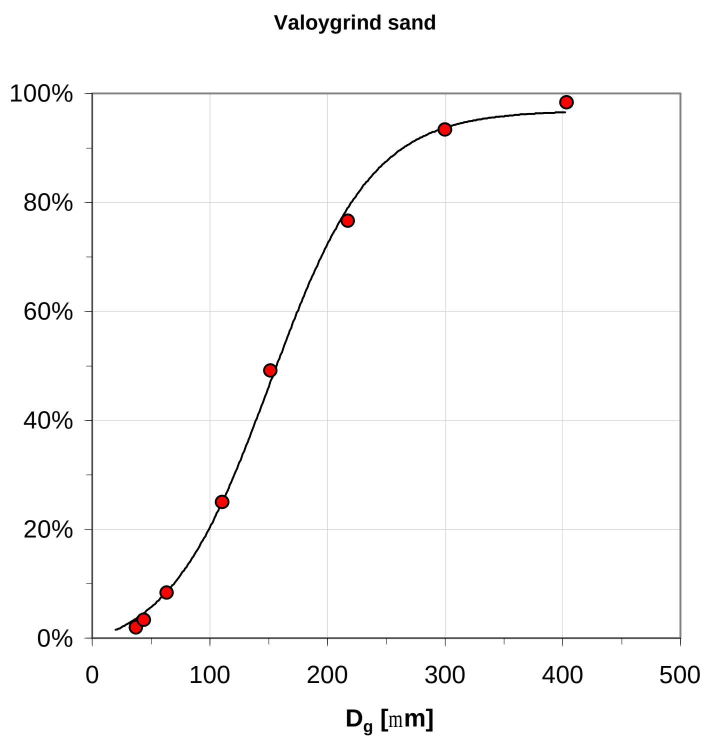
1.3 Exercise

The sieve curve of a particular sand is given within good approximation by the so-called logistic curve,

$$N \approx \frac{1}{1 + C\exp(-D/D_1)} - N_0$$

where $N_0 = 0.031,\ C = 32.7\ D_1 = 43.4\ \mu m$.

Determine the density distribution of a water column at times 1sec and 5 sec and after the end of the mixing process.



Given are the density of grains, $\rho_S = 2.65\,gr/cm^3$ and the initial solids concentration $c_0 = 0.1$.

2. MOTION

2.1 Description of Motion

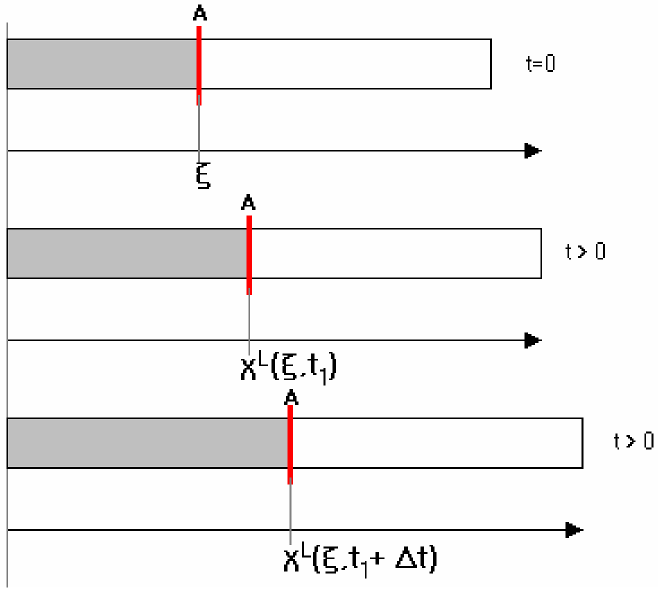
For the description of a continuous medium we need the concept of *material point*. To each material point we assigned a point P in space, e.g. the center of gravity of the corresponding REV. As illustrated in the previous chapter, the most *primitive* mechanical property assigned to a material point is the *mass density* of the continuum at that point. The second important primitive property assigned to a material point is that of its *velocity*. We may again imagine that the velocity of the material point is some kind of average velocity of the microscopic particles that occupy the corresponding REV. This theoretical basis for the study of motion of deformable bodies is attributed again to Euler, who studied the motion of the material points of a fluid along the so-called *streamlines.* The streamlines of the fluid particles constitute a double infinity of curves in space with the following property: The particle velocity vector at a given point in space and time is tangential to the streamline that is passing through that point at that time. Based on the work of Euler, Lagrange[1] observed that since we are dealing with a triple infinity of particles, in some occasions it is meaningful to consider instead of the streamlines the so-called *pathlines,* i.e. the triple infinity of particle paths in space. Accordingly in Continuum Mechanics we distinguish among the two viewpoints, the *"lagrangian"* and the *"eulerian"* description of motion.

2.2 Lagrangian or Material Description

At a given time $t=0$ the material points of a continuum occupy certain positions in space, which are easily described by the Cartesian co-ordinates of these points, with respect to a fixed-in-space coordinate system. For

[1] Louis de Lagrange 1736-1813.

example for a one-dimensional continuum (e.g. for a rod) the material point **A** (the cross-section of the rod) takes at time $t=0$ the position $x=\xi$. At another time $t\geq 0$, the same material point A takes the position $x = x^L$. The superscript L denotes that the particular description of the motion of the continuum is the after Lagrange that pays attention to movement of the particles themselves.



This description is also called material description of the motion. Accordingly we assume that there exists a function

$$x^L = \chi^L(\xi,t) \tag{2.1}$$

which for all t describes the **motion** of the considered body. With the assumptions made above, the spatial co-ordinates at time $t=0$ mark the "names" of the material points

$$\chi^L(\xi,0) = \xi$$

The **displacement** of the material point is defined as the difference of its position (vectors[2]) at times t and $t=0$, respectively,

$$u^L(\xi,t) = x - \xi = \chi^L(\xi,t) - \xi$$

Let two neighboring positions of the material point **A** at times t_1 and $t_2 = t_1 + \Delta t$,

$$x_{(1)} = \chi^L(\xi,t_1) \quad , \quad x_{(2)} = \chi^L(\xi,t_2)$$

[2] In 2 and 3D the position, displacement, velocity and acceleration of the particles are vector quantities.

We see in the following that the corresponding *incremental displacement* is computed as the finite difference of these two neighboring positions,

$$x_{(2)} = \chi^L(\xi, t_1 + \Delta t) \approx \chi^L(\xi, t_1) + \left.\frac{\partial \chi^L}{\partial t}\right|_{\xi} \Delta t \quad \Rightarrow \quad \Delta u^L = u^L(\xi, t_2) - u^L(\xi, t_1) \approx \left.\frac{\partial \chi^L}{\partial t}\right|_{\xi} \Delta t$$

With that we are able to compute the **velocity** of the material point at time t,

$$v^L(\xi, t) = \lim_{\Delta t \to 0}\left(\frac{\Delta u^L}{\Delta t}\right) = \left.\frac{\partial \chi^L}{\partial t}\right|_{\xi}$$

Similarly we define the **acceleration** of the material point,

$$a^L(\xi, t) = \left.\frac{\partial v^L}{\partial t}\right|_{\xi} = \left.\frac{\partial^2 \chi^L}{\partial t^2}\right|_{\xi}$$

The 'particle' velocity and 'particle' acceleration signify changes in mechanical properties of the material points, namely changes in their position and velocity, respectively. Thus the partial time derivative of a quantity

$$\phi = \varphi^L(\xi, t)$$

is called the **material time derivative** and is denoted as

$$\frac{D\phi}{Dt} = \left.\frac{\partial \varphi^L(\xi, t)}{\partial t}\right|_{\xi = \text{const.}}$$

Sometimes the material time derivative of a quantity $\phi = \varphi^L(\xi, t)$ is denoted by a superimposed dot, and is called the **rate** of ϕ

$$\dot{\phi} \equiv \frac{D\phi}{Dt}$$

Thus velocity and acceleration will by expressed as,

$$v = \dot{x} = \frac{Dx}{Dt} = \left.\frac{\partial \chi^L(\xi, t)}{\partial t}\right|_{\xi = \text{const.}}$$

$$a = \dot{v} = \frac{Dv}{Dt} = \left.\frac{\partial v^L(\xi, t)}{\partial t}\right|_{\xi = \text{const.}}$$

2.3 Eulerian or Spatial Description

For an Eulerian description all quantities are seen as functions of the spatial co-ordinates of the material points at current time. Thus we assume that the equation of motion (2.1) is uniquely invertible, resulting into a functional dependency of the material co-ordinate to the current position of the material point,

$$\xi = \chi^E(x,t)$$

with the property,

$$x = \chi^L(\xi,t) = \chi^L\left(\chi^E(x,t),t\right) \quad \forall t$$

$$\xi = \chi^E(x,t) = \chi^E\left(\chi^L(\xi,t),t\right) \quad \forall t$$

Similarly the displacement can be written in Eulerian co-ordinates as,

$$u = u^L(\xi,t) = u^L\left(\chi^E(x,t),t\right) = u^E(x,t)$$

and accordingly,

$$x = \xi + u \quad \Rightarrow \quad \chi^E(x,t) = x - u^E(x,t)$$

Example[3]

Let as assume for example that the material points of a 1D-continuum are moving according to the following law

$$x = \chi^L(\xi,t) = \xi\left(1 + \frac{1}{2}\left(\frac{t}{t_c}\right)^2\right) \quad , \quad t_c = \text{const.}$$

$$\Leftrightarrow$$

$$\xi = \chi^E(x,t) = \frac{x}{1 + \frac{1}{2}\left(\frac{t}{t_c}\right)^2}$$

We observe that for $t=0$, $x=\xi$, which means that the initial position of the particles is assumed to be the reference position for the material description of the motion.

The displacement in Lagrangian description is

[3] cf. A.J. Roberts, *A One-Dimensional Introduction to Continuum Mechanics*, World Scientific, 1994.

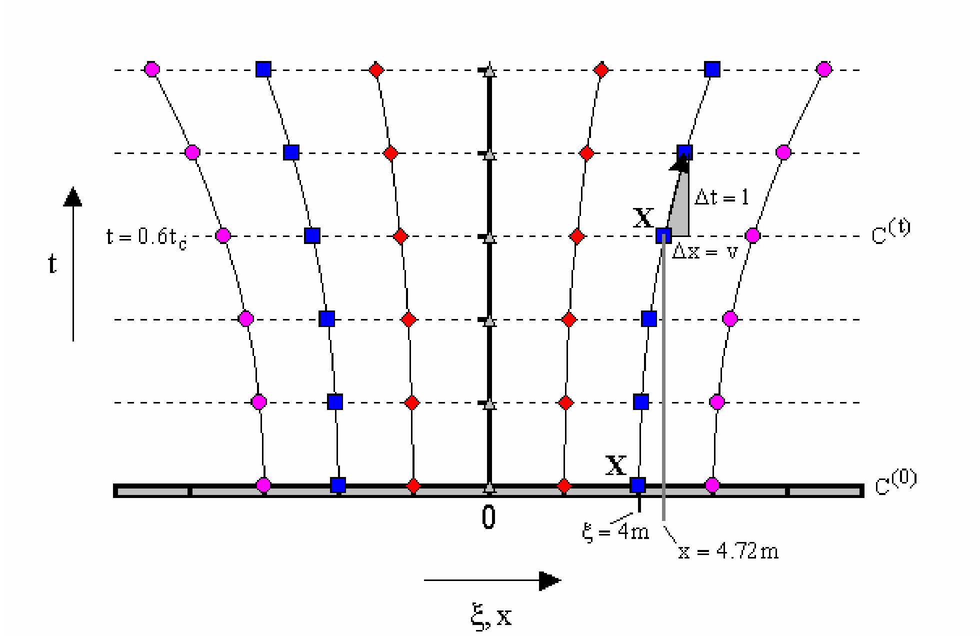
$$u^L = u(\xi,t) = \frac{1}{2}\xi\left(\frac{t}{t_c}\right)^2$$

The motion of an 1D continuum is sometimes represented graphically in the 2D space-time plane ? (x,t). This space is called the space of **events**, since any pair (x,t) in that space will be called an event . The curve

$$x=\chi^L(\xi,t)$$

in space-time is called the **lifeline** of the corresponding particle, $X(\xi)$. This means in turn that from the axis $t=0$ will start the lifelines of the various particles, which at time $t=0$ occupied the position $x=\xi$ (reference configuration $C^{(0)}$). The current configuration $C^{(t)}$ of the particles at any given time $t \geq 0$, is seen on horizontal lines with $t = \text{const.}$

In the considered example all lifelines are diverging parabolas, which means in turn that with time the relative distances of the particles increase. Such a motion is called a *dilation*.



With the motion of the particles being defined, we can easily compute their velocity as function of time and their position in the reference configuration,

$$v = v^L(\xi,t) = \frac{\partial \chi^L}{\partial t} = \frac{\partial u^L}{\partial t} = \xi\frac{t}{t_c^2}$$

Following the above definitions, we may see the velocity of the particle at any time as function of its position in the current configuration,

$$v = v^L\left(\chi^E(x,t),t\right) = \frac{x}{1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2}\frac{t}{t_c^2} \quad \Rightarrow$$

$$v = v^E(x,t) = \frac{\frac{t}{t_c}}{1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2}\frac{x}{t_c}$$

As already said in this example the reference configuration $C^{(0)}$ coincides with the x-axis $(t=0)$, where we pointed out as an example point X at position $\xi = 4m$. At time $t = 0.6t_c$ the current configuration $C^{(t)}$ of the considered 1D-continuum is along the axis $t = 0.6\,t_c$ and the material point X occupies the position $x = 4.72m$. With $t_c = 1.s$, in the considered example we get,

$$v^L(4.m,0.6s) = \ \xi\frac{t}{t_c^2} = \frac{4m}{1s}0.6 = 2.4\frac{m}{s}$$

$$v^E(4.72m,0.6s) = \frac{0.6}{1+\frac{1}{2}0.6^2}\cdot\frac{4.72m}{1s} = 0.508475\cdot 4.72\frac{m}{s} = 2.4\frac{m}{s}.$$

Finally we remark that the particle velocity appears in space-time as the inverse slope of the corresponding lifeline at the considered event.

2.4 Material Time Derivative in Eulerian Description

As already mentioned the material time derivative of a quantity $\phi = \varphi^L(\xi,t)$ is denoted as

$$\dot{\phi} = \frac{D\phi}{Dt} = \frac{\partial}{\partial t}\varphi^L(\xi,t)$$

The material time derivative is juxtaposed from the **local time derivative** of a quantity $\phi = \varphi^E(x,t)$ **,**

$$\frac{\delta\phi}{\delta t} = \frac{\partial}{\partial t}\varphi^E(x,t)$$

The material time derivative can be computed directly from the Eulerian description of the considered field as follows: Let

$$\Delta\phi = \varphi^E(\bar{x},\bar{t}) - \varphi^E(x,t)$$

$$\bar{x} = x + \Delta x = x + v^E(x,t)\Delta t$$

$$\bar{t} = t + \Delta t$$

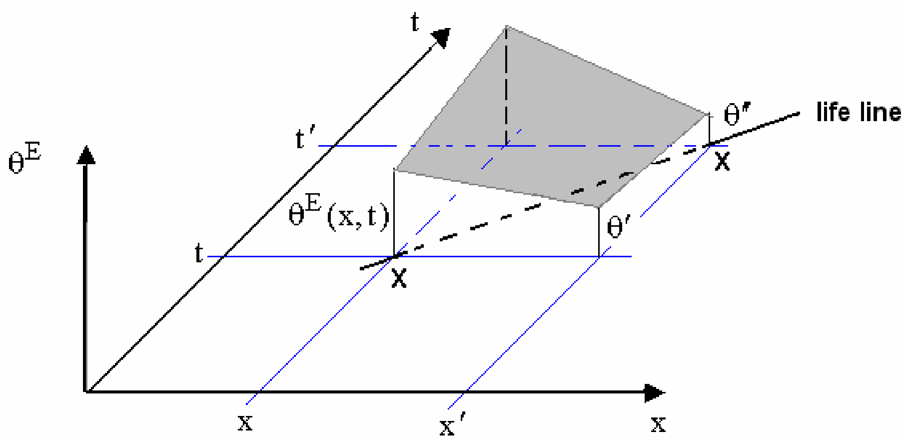
and thus,

$$\varphi^E(\bar{x},t) \approx \varphi^E(x,t) + \frac{\partial}{\partial x}\varphi^E(x_i,t)\,\Delta x$$

$$\varphi^E(\bar{x},\bar{t}) \approx \varphi^E(\bar{x},t) + \frac{\partial}{\partial t}\varphi^E(\bar{x},t)\,\Delta t$$

$$\approx \varphi^E(x,t) + \frac{\partial}{\partial x}\varphi^E(x,t)\,\Delta x + \frac{\partial}{\partial t}\left(\varphi^E(x,t) + \frac{\partial}{\partial x}\varphi^E(x,t)\,\Delta x\right)\Delta t$$

$$\Rightarrow \quad \Delta\phi = \varphi^E(\bar{x},\bar{t}) - \varphi^E(x,t) = \frac{\partial}{\partial t}\varphi^E(x,t)\,\Delta t + \frac{\partial}{\partial x}\varphi^E(x,t)\,\Delta x + O(\Delta x \cdot \Delta t)$$



With the remark that along the lifeline of a particle,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

we get finally the formula for the computation of the material time derivative within the frame of a spatial description of the motion,

$$\frac{D\phi}{Dt} = \frac{\partial \varphi^E}{\partial t} + v^E \frac{\partial \varphi^E}{\partial x} \qquad (2.2)$$

or in brief

$$\frac{D\phi}{Dt} = \frac{\delta\phi}{\delta t} + v\frac{\partial \phi}{\partial x}$$

Remark:

In multidimensional problems we have the following formulae for the material time derivative

$$\dot{\phi} = \frac{D\phi}{Dt} = \frac{\partial \varphi^E}{\partial t} + \sum_{k=1}^{3} v_k \frac{\partial \varphi^E}{\partial x_k}$$

or

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \mathrm{grad}(\phi)$$

In these expression the term $\frac{\partial \varphi^E}{\partial t}$ is called **local** and the term $v^E \frac{\partial \varphi^E}{\partial x}$ is called **convective**. We remark that when the velocity of a material point as well as its gradient are infinitesimal then the contribution of the convective term may be negligible, and

$$\dot{\phi} \approx \frac{\partial \varphi^E}{\partial t}$$

The acceleration a is defined as the rate of the velocity of material point. Thus have the following equivalent expressions,

$$a = a^L(\xi\ \ ,t) = \frac{\partial}{\partial t} v^L(\xi\ \ ,t) \quad \Rightarrow$$

$$a = a^E(x,t) = \frac{D}{Dt} v^E(x,t) = \frac{\partial v^E}{\partial t} + v\frac{\partial v^E}{\partial x}$$

Example

In the considered example we compute the acceleration either by using the Lagrangian description of the velocity

$$v^L = \xi \frac{t}{t_c^2} \quad \Rightarrow \quad a^L = \frac{\xi}{t_c^2}$$

or the Eulerian description,

$$a^E = \frac{\frac{1}{t_c}\left(1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2\right) - \frac{t}{t_c}\frac{t}{t_c^2}}{\left(1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2\right)^2}\frac{x}{t_c} + \frac{\frac{t}{t_c}}{1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2}\frac{x}{t_c}\frac{\frac{t}{t_c}}{1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2}\frac{1}{t_c}$$

$$a^E = \frac{\frac{1}{t_c}}{\left(1+\frac{1}{2}\left(\frac{t}{t_c}\right)^2\right)}\frac{x}{t_c} = \frac{\xi}{t_c^2} = a^L$$

2.5 Exercise

A one-dimensional continuum is deforming according to the Lagrangian description

$$x = \chi^L(\xi, t) = \frac{\xi}{1+\lambda\xi t}$$

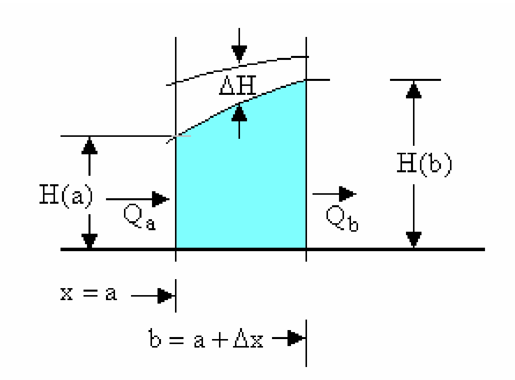
1. Find the Lagrangian description of the velocity and acceleration.
2. Find the Eulerian description of he motion
3. Find the Eulerian description of the velocity and acceleration using the formula (2.2) for the material time derivative.

3. MASS BALANCE

3.1 Introductory Examples

3.1.1 Mass storage: The open channel flow

In order to introduce the concept of mass balance we consider as a first example the flow of water in an open channel. Accordingly we define the following quantities:



- $\rho = const.$: the density of water[1]
- $B = const.$: the width of the channel
- $H(x,t)$: the height of water along the channel
- $v(x,t)$: the height-averaged fluid velocity
- $Q = Hv$: the fluid-discharge[2]
- $\dot{m} = \rho BQ$: the total water mass-flow across a vertical section

[1] In problems with a **free surface** the aqueous phase is considered as incompressible.
[2] Q is measured for example in m^3/s per running meter in the direction of the width of the channel.

The mass-inflow and outflow of water for a **control volume** between two given sections at $x = a$ and $x = b$ is

$$\dot{m}_{in} = \rho BQ(a,t) \quad , \quad \dot{m}_{out} = \rho BQ(b,t)$$

Conservation of mass for the considered control volume yields the well-known **storage equation** as follows: Let Δm_Q be the **net mass-influx** of water during the time interval Δt between two **neighboring** sections at $x = a$ and $x = b = a + \Delta x$:

$$\begin{aligned}&\Delta m_Q = (\dot{m}_{in} - \dot{m}_{out})\Delta t = \left(\rho BQ(x,t) - \rho BQ(x+\Delta x,t)\right)\Delta t\\&\Rightarrow\\&\Delta m_Q = -\rho B\Delta Q\Delta t \quad , \quad \Delta Q = Q(x+\Delta x,t) - Q(x,t) \approx \left(\frac{\partial Q}{\partial x}\right)_{x,t}\Delta x\end{aligned} \tag{3.1}$$

Let on the other hand Δm_H be the water mass **stored** inside this control volume. Mass storage during the considered time interval will cause an elevation of the free water surface by ΔH. Thus the mass stored inside the control volume is computed as

$$\begin{aligned}&\Delta m_H = \rho B(b-a)\Delta H = \rho B\Delta x\left(H(x,t+\Delta t) - H(x,t)\right)\\&\Rightarrow\\&\Delta m_H = \rho B\Delta H\Delta x \quad , \quad \Delta H = H(x,t+\Delta t) - H(x,t) \approx \left(\frac{\partial H}{\partial t}\right)_{x,t}\Delta t\end{aligned} \tag{3.2}$$

Since no mass is generated or produced inside the considered control volume, mass balance

$$\Delta m_Q = \Delta m_H \tag{3.3}$$

is expressing mass conservation. Above mass balance equation (3.3) together with equations (3.1) and (3.2) gives the well- known storage equation of open channel flow,

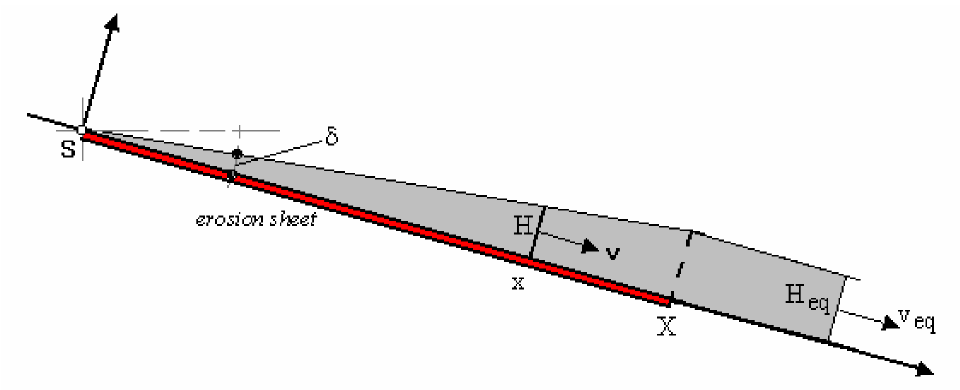
$$\underline{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(Hv) = 0} \tag{3.4}$$

3.1.2 Mass generation: The erosion sheet

As a second example we consider the steady-state solution of the conservation equation (3.4),

$$Hv = Q_\ell = \text{const.} \quad \Rightarrow \quad v = \frac{Q_\ell}{H} \tag{3.5}$$

and apply this solution to a particular geologic structure, the so-called **erosion sheet.** This is a wedge-shaped creeping landslide, whose thickness increases linearly with the runoff distance x from the source S.



This means that we try a solution for the landslide height of the form

$$H = \tan\delta \cdot x \quad (3.6)$$

This solution provides a singularity for the velocity at the source S, since

$$v = \frac{Q_\ell}{\tan\delta}\frac{1}{x} \quad (3.7)$$

This velocity singularity means that the considered solution, given by equation (3.7), is physically questionable in the root region of the landslide. This observation suggests in turn to search for an appropriate **regularization** of the singular solution. As we will see below this regularization can be achieved by modifying the mass balance equation so as to account for erosion and deposition.

Indeed we may assume that in the considered transitional region mass storage consists of two terms, one that accounts for the changes in height and another that accounts for net material erosion,

$$\Delta m_{st} = \Delta m_H - \Delta m_{er} \quad (3.8)$$

This assumption modifies the mass balance equation (3.3) to the following,

$$\Delta m_Q = \Delta m_H - \Delta m_{er} \quad (3.9)$$

It is convenient to express the rate of eroded mass per unit length of the erosion sheet in terms of a quantity s_{er}, which has the dimension of velocity, and is accordingly called the **erosion speed**:

$$\dot{m}_{er} = \rho B s_{er} \quad \Rightarrow \quad \Delta m_{er} = \rho B s_{er} \Delta t \Delta x \quad (3.10)$$

With this notation, above mass balance equation (3.9) yields,

$$-\rho B\left(\frac{\partial Q}{\partial x}\right)\Delta x\Delta t = \rho B\left(\frac{\partial H}{\partial t}\right)\Delta t\Delta x - \rho B s_{er}\Delta t\Delta x \quad \Rightarrow$$

$$\underline{\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(Hv) = s_{er}} \tag{3.11}$$

From dimensional analysis considerations we conclude that the erosion speed appearing on the r.h.s. of the mass balance equation (3.11) is a function of the flow-velocity and of the equilibrium velocity,

$$s_{er} = F(v, v_{eq}) \approx c_1 v + c_2 \frac{v^2}{v_{eq}} + \cdots$$

We may for example assume that in regions of high velocities erosion dominates, whereas in regions of low velocities deposition dominates. Accordingly we may consider the following **constitutive model** for the erosion speed,

$$s_{er} = -e\left(v - \frac{v^2}{v_{eq}}\right) \qquad (e = \text{const.} > 0) \tag{3.12}$$

The constitutive equation (3.12) of mass erosion-deposition contains two parameters, the erosion constant, e, and the equilibrium velocity v_{eq}. The equilibrium velocity is the one of the steadily creeping landslide. This means that erosion and deposition ($\dot{m}_{er} > 0$) are taking place in a region $0 < x < X$, close to source S of the landslide, where the slide velocity is high, $v > v_{eq}$. Past that distance X, the landslide is having practically constant height $H_{eq} = Q_\ell / v_{eq}$ and is creeping steadily with the equilibrium velocity v_{eq}.

The above mass balance and mass erosion equations (3.11) and (3.12) respectively, result finally to the following mass-balance equation with mass **'generation-annihilation'** terms,

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(Hv) = -e\left(v - \frac{v^2}{v_{eq}}\right) \tag{3.13}$$

By inserting in equation (3.13) the erosion sheet solution, equation (3.6)

$$H = \tan\delta\ x \quad \Rightarrow \quad \frac{\partial}{\partial t}H(x,t) = 0\,,\ \frac{\partial}{\partial x}H(x,t) = \tan\delta$$

we obtain an ordinary differential equation for the velocity,

$$x\frac{dv}{dx} + (1+\varepsilon)v - \varepsilon\frac{v^2}{v_{eq}} = 0 \qquad (3.14)$$

In order to treat further the governing differential equation (3.14) we introduce a dimensionless velocity and a dimensionless position coordinate

$$\tilde{v} = \frac{v}{v_{eq}} \quad , \quad \tilde{x} = \frac{x}{X} \quad (0 \le \tilde{x} \le 1)$$

X is the *runoff distance* of the sheet from the source at which ,

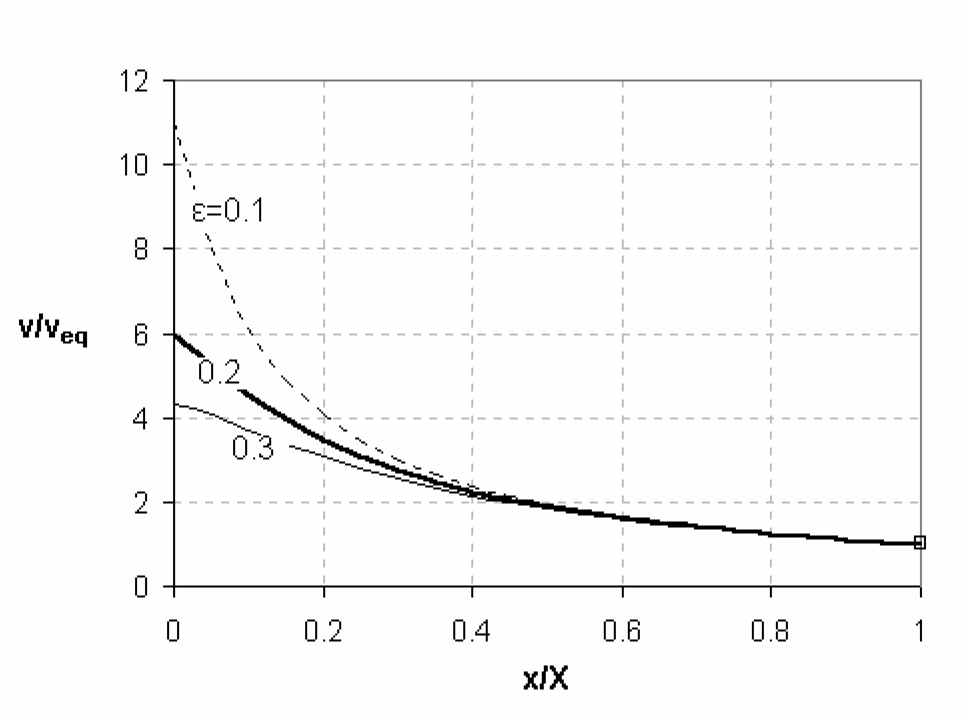
$$v(X) = v_{eq} \quad \Rightarrow \quad \tilde{v}(1) = 1$$

In terms of dimensionless quantities, the governing equation (3.14) becomes a Riccati-type o.d.e.

$$\tilde{x}\tilde{v}' + (1+\varepsilon)\tilde{v} - \varepsilon\tilde{v}^2 = 0 \qquad (3.14\text{bis})$$

The solution of equation (3.14bis) reads then,

$$\tilde{v} = \frac{1}{\tilde{x}^{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}\left(1 - \tilde{x}^{1+\varepsilon}\right)} \quad ; \quad \tilde{x} \ni [0,1] \qquad (3.15)$$



We observe that the consideration of erosion/deposition terms in the mass balance equation has the expected regularizing effect on the velocity distribution for the erosion sheet solution, since for

$$\varepsilon > 0 \ , \ x \to 0^{+} \qquad : \qquad v \to v_0^{+} = \left(1 + \frac{1}{\varepsilon}\right) v_{eq} < \infty$$

the velocity at the source remains finite. This solution suggests also that at the source S $(x = 0)$ a steadily opening crack will form, since the velocity is discontinuous at that point,

$$[\![v]\!]_S = v(0^{+}) - v(0^{-}) = v_0^{+}$$

Thus the opening rate of the crack at the source is essentially reflecting the effect of the erosion parameter $\varepsilon^{-1} = \tan\delta / e$.

3.2 The Mass Balance Equation

We consider now one-dimensional flow of mass along a tube and we focus our attention in the region between two sections at finite distance, with the Lagrangian co-ordinates α and β, respectively. At time t these sections have the spatial coordinates

$$a = \chi^{L}(\alpha, t) = \hat{a}(t) \quad , \quad b = \chi^{L}(\beta, t) = \hat{b}(t) \tag{3.16}$$

The mass that is contained inside the volume between these two sections is

$$m(t) = \int_a^b \rho(x,t)\, dx \tag{3.17}$$

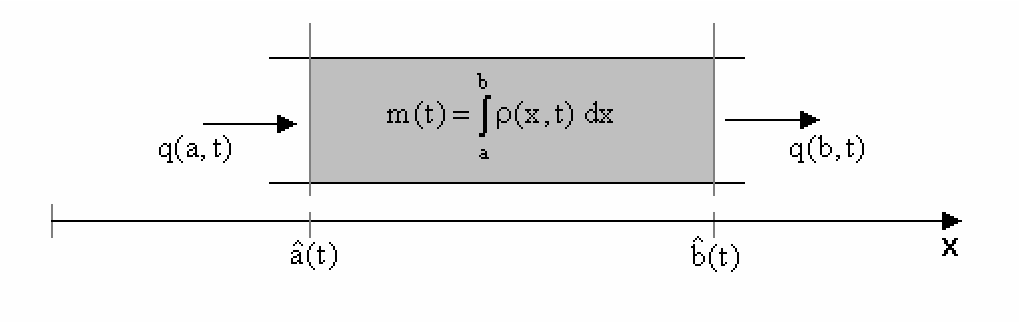
In order to compute the rate of mass

$$\dot{m} = \frac{dm}{dt}$$

we use the well-known differentiation formula of integrals with variable integration limits[3],

$$\dot{m} = \frac{d}{dt} \int_{\hat{a}(t)}^{\hat{b}(t)} \rho(x,t) dx = \int_{\hat{a}(t)}^{\hat{b}(t)} \frac{\partial}{\partial t} \rho(x,t) dx + \frac{d\hat{b}}{dt} \rho(b,t) - \frac{d\hat{a}}{dt} \rho(a,t) \tag{3.18}$$

[3] In the Fluid Mechanics Literature what follows is known as ***Reynolds's Transport Theorem***, especially when the derivations are generalized in 3D via Gauss' divergence theorem.



From equations (3.16) we get,

$$\frac{d\hat{a}}{dt} = \frac{\partial \chi^L(\alpha,t)}{\partial t} = v^L(\alpha,t) = v^E(a,t) = v(a,t)$$

$$\frac{d\hat{b}}{dt} = \frac{\partial \chi^L(\beta,t)}{\partial t} = v^L(\beta,t) = v^E(b,t) = v(b,t)$$

and with that equation (3.18) becomes

$$\dot{m} = \frac{d}{dt}\int_{\hat{a}(t)}^{\hat{b}(t)} \rho(x,t)dx = \int_a^b \frac{\partial}{\partial t}\rho(x,t)dx + v(b,t)\rho(b,t) - v(a,t)\rho(a,t) \qquad (3.19)$$

We define the **mass-flux**

$$q = \rho\, v \qquad (3.20)$$

and observe that eq. (3.19) reads as

$$\dot{m} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\rho(x,t)dx + q(b,t) - q(a,t) \qquad (3.21)$$

This means that changes in mass inside the control volume may by due to two effects,

1. The density of the substance inside the control volume is changing

2. The inflow $q(a,t)$ and the outflow $q(b,t)$ do not balance each other.

Summarizing the above results we have the following expression for the rate of mass in a given control volume between two sections,

$$\dot{m} = \frac{d}{dt}\int_{a(t)}^{b(t)} \rho(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\rho(x,t)dx + v(b,t)\rho(b,t) - v(a,t)\rho(a,t)$$

or

$$\dot{m} = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}\rho(x,t)dx + [\rho v]_{a(t)}^{b(t)} = \int_{a(t)}^{b(t)} \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right]dx \qquad (3.22)$$

We assume now that in the considered control volume mass is neither produced nor removed, then mass is **conserved**, and

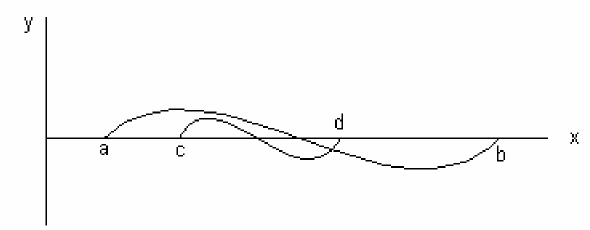
$$\dot{m} = \frac{dm}{dt} = 0 \qquad (3.23)$$

resulting into the following **global** (or weak) form of the mass conservation law,

$$\int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right] dx = 0 \qquad (3.23a)$$

In order to arrive in a local form[4] of the mass conservation law we resort to the following theorem of Analysis:

Theorem: Let $f(x)$ be a **continuous** function in the $[c,d]$ and let also $\int_a^b f(x)\,dx = 0 \;\; \forall a,b$ s.t. $c < a < b < d$, then $f(x) = 0 \;\forall\, x \in [c,d]$.



Thus if we assume that inside the considered control volume **no discontinuities** of the density function exist, then

$$\int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right] dx = 0 \quad \forall [a,b] \quad \Leftrightarrow \quad \underline{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0} \quad \forall x \in [a,b] \qquad (3.24)$$

By utilizing the definition of material time derivative,

$$\dot{\rho} = \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}$$

the *localized* form of mass conservation may be written as,

[4] Sometimes this local form is called the *localization* of the global intergral equation.

$$\frac{D\rho}{Dt} + \rho \frac{\partial v}{\partial x} = 0 \qquad (3.25)$$

Remark:

In analogy to eq. (3.24), in multidimensional problems the mass conservation law is expressed as

$$\underline{\dot{\rho} + \rho \operatorname{div} \bar{v} = 0}$$

Exercise

Derive the storage equation (3.4) by utilizing the mass conservation equation (3.24) and by identifying the density in that equation with the 'linear' density

$$\rho_{\ell} = \rho B H(x,t)$$

3.3 The Continuity Equation

In most instances water flow in closed pipes is characterized by constant density[5],

$$\rho = \rho_0 = \text{const}$$

Introducing again the mass flux,

$$q = \rho\, v(x,t)$$

from equation (3.24) we obtain

$$\rho_0 \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(\rho_0 v) = \frac{\partial q}{\partial x} = 0$$

or

$$\int_a^b \frac{\partial q}{\partial x} dx = 0 \quad \Rightarrow \quad q(b,t) = q(a,t)$$

The equation:

$$\frac{\partial q}{\partial x} = 0 \quad \Leftrightarrow \quad q(b,t) = q(a,t) \qquad (3.26)$$

is known as the **continuity equation** for incompressible fluids.

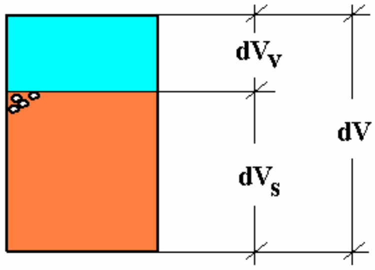
[5] A counter example is the so-called *water-hammer wave* that takes place when a valve is closed suddenly in a pipe with flowing water. A water-hammer wave can generate enormous pressure due to the extremely low compressibility of water.

3.4 Mass Balance in Porous Media

Consider an REV of a porous medium with total volume dV, consisting of:

a) solid grains with partial volume dV_S and mass dm_S *(s: solids)*

b) voids with partial volume dV_V and zero mass *(v: voids)*.



The void's ratio e and the porosity[6] ϕ of the porous medium are defined as

$$e = \frac{dV_V}{dV_S} \quad , \quad \phi = \frac{dV_V}{dV} = \frac{dV_V}{dV_S + dV_V} = \frac{e}{1+e} \tag{3.27}$$

Let

$$\rho_S = \frac{dm_S}{dV_S} \tag{3.28}$$

be the density of the grains. The **partial density** of the porous medium is computed as follows

$$\rho_1 = \frac{dM_S}{dV} = \frac{dM_S}{dV_V + dV_S} = \frac{1}{1+e}\frac{dM_S}{dV_S} = (1-\phi)\rho_S \qquad (3.28)$$

Considering again an 1D-deformation process, let

$$v^{(1)} = v(x,t)$$

be the velocity of the particles of the corresponding continuum.

We assume now that the grains are practically incompressible

$$\rho_S = \text{const.}$$

Under these definitions and assumptions mass balance, as expressed by equations (3.24) or (3.25), yields,

---

[6] In Soil Mechanics literature the porosity is denoted by the symbol n.

$$\frac{D}{Dt}\left((1-\phi)\rho_s\right)+(1-\phi)\rho_s\frac{\partial v^{(1)}}{\partial x}=0$$

or

$$-\dot{\phi}+(1-\phi)\frac{\partial v^{(1)}}{\partial x}=0 \tag{3.29}$$

or

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi v^{(1)})=\frac{\partial v^{(1)}}{\partial x} \tag{3.30}$$

Let also

$$\dot{\varepsilon}=\frac{\partial v^{(1)}}{\partial x} \tag{3.31}$$

denote the volumetric strain-rate. From the mass balance equation (3.29), we recover the well-known formula of soil mechanics

$$\dot{\varepsilon}=\frac{1}{1-\phi}\dot{\phi} \tag{3.32}$$

3.5 The Rankine-Hugoniot Condition for Density Shocks

The "*localizations"* of the mass balance law, eqs. (3.24) or (3.25), are valid only if the density is continuous in the considered interval. In case where the density is discontinuous, some restriction or *'compatibility condition'* is derived from the global conservation law eq. (3.23). This restriction applies to the 'speed' of propagation of such density discontinuity. For this purpose we consider the case where at a position

$$x=D(t) \tag{3.33}$$

a density discontinuity[7] (D: discontinuity) is occurring. We define the so-called density **jump**

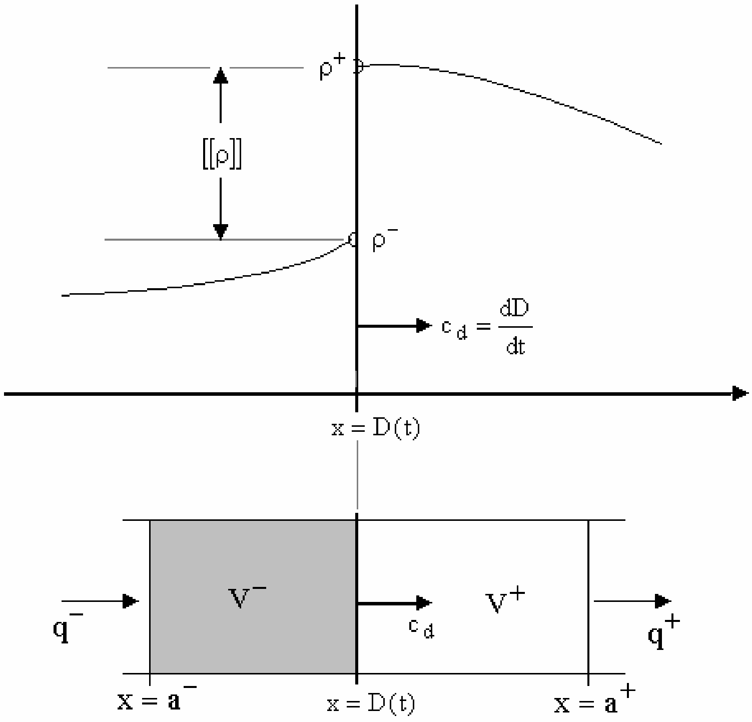
$$[\![\rho]\!]=\rho^{+}-\rho^{-}=\rho\left(D^{+},t\right)-\rho\left(D^{-},t\right) \tag{3.34}$$

Across the discontinuity D the velocity may also be discontinuous. Thus we define the jump in mass flux as

$$[\![q]\!]=q^{+}-q^{-}=q\left(D^{+},t\right)-q\left(D^{-},t\right)$$

[7] D is called a *strong* discontinuity, because across D the density itself is discontinuous.

$$[[q]] = \rho^+ v^+ - \rho^- v^- \quad (3.35)$$



Let

$$c_d = \frac{dD}{dt} \tag{3.36}$$

be the speed of propagation of the discontinuity[8] . Since mass is conserved on both sides of the discontinuity, we get,

$$\dot{m}^- = \int_{a^-}^{D(t)} \frac{\partial \rho}{\partial t} dx + \rho^- c_d - q(a^-)$$

$$\dot{m}^+ = \int_{D(t)}^{a^+} \frac{\partial \rho}{\partial t} dx + q(a^+) - \rho^+ c_d$$

Thus

$$\dot{m} = \dot{m}^+ + \dot{m}^- = \int_{D(t)}^{a^+} \frac{\partial \rho}{\partial t} dx + q(a^+) - \rho^+ c_d + \left(\int_{a^-}^{D(t)} \frac{\partial \rho}{\partial t} dx + \rho^- c_d - q(a^-) \right)$$

[8] D is called also a *shock wave.*

or

$$\dot{m} = \int_{D(t)}^{a^+} \frac{\partial \rho}{\partial t} dx + \int_{a^-}^{D(t)} \frac{\partial \rho}{\partial t} dx + q(a^+) - \rho^+ c_d + \left(\rho^- c_d - q(a^-) \right)$$

Mass conservation requires that

$$\dot{m} = 0 \quad \Rightarrow \quad \int_{a^-}^{a^+} \frac{\partial \rho}{\partial t} dx + q(a^+) - \rho^+ c_d - \left(q(a^-) - \rho^- c_d \right) = 0$$

If take now the limit

$$a^+ = -a^- = a \to 0$$

the integral in the above expression is not contributing, and mass conservation across the discontinuity results into the so-called **Rankine-Hug?niot condition**

$$q^+ - \rho^+ c_d = q^- - \rho^- c_d \quad \Rightarrow \quad \rho^+ v^+ - \rho^+ c_d = \rho^- v^- - \rho^- c_d$$

or

$$\rho^+ \left(v^+ - c_d \right) = \rho^- \left(v^- - c_d \right) \tag{3.37}$$

which connects the density jump and flux jump to the propagation velocity of the shock D(t),

$$c_d = \frac{[[q]]}{[[\rho]]} \tag{3.38}$$

3.6 Porosity Shocks in the Fluidized-Column Test [9]

The fluidized column experiment is used to study the phenomena of fluid-flow in granular media under relatively high hydraulic gradients, which act opposite to gravity. This test has been used extensively to analyze filtering processes and make assessments about internal stability of sand filters.

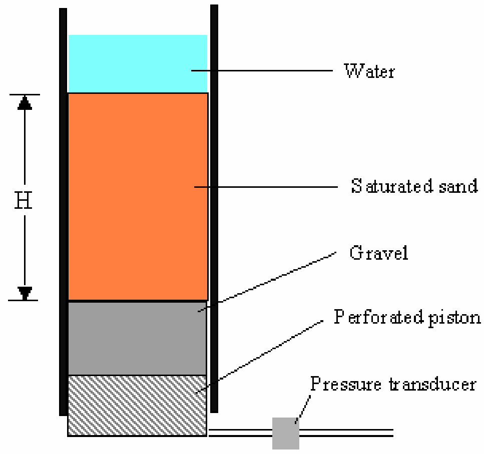
As is well known from soil mechanics literature, at a critical hydraulic gradient J_c, the seepage force excreted on the grains by the flowing fluid in upwards

[9] Vardoulakis, I, Stavropoulou, M. and Skjaerstein, A. (1998). Porosity waves in fluidized sand-column tests. *Proc. Royal Soc. London A.,* **356**, 1-18.
Vardoulakis, I. (2002). Fluidization in artesian flow conditions: I. Hydro-mechanically stable granular media. Géotechnique, submitted for publication.
Vardoulakis, I. (2002). Fluidization in artesian flow conditions: II. Hydro-mechanically unstable granular media. Géotechnique, submitted for publication

flow balances their buoyant weight. The critical hydraulic gradient has the value

$$J_c = (1-\phi)\left(\frac{\rho_s}{\rho_f} - 1\right)$$

where ρ_s and ρ_f are the grain and fluid densities, and ϕ the porosity of the granular assembly[10]. At and past this critical hydraulic gradient, '*quick-sand'* conditions are holding, which are characterized by relative mobility of the sand grains due to total loss of their frictional strength. Due to pronounced internal grain mobility, quick-sand behaves more like a 'compressible' fluid with remarkable density fluctuations.



Fluidized-column experiments have shown that changes in flow-rate alter the porosity of quick-sand. CT-images taken at constant flow-rate in the transient regime show a process, which could be described as a 'density wave' motion. A change in the flow-rate gives rise to an upwards propagating density wave in the sand body. When the wave reaches the top of the sand body a new equilibrium is reached. These tests indicated also that the amplitude of the density wave is related to the change in the flow rate.

Following the CT observations we consider a simple wave model for the porosity change due to sudden changes in flow conditions. In that sense we assume that at the instant of sudden application of any new level of flow-rate, a planar porosity shock-wave moves upwards with speed c_d. We assume that in the front of the wave the porosity is equal to the initial porosity and in the back it is equal to its final value

$$\phi^+ = \phi_{int} \quad , \quad \phi^- = \phi_{fnl}$$

The experiment has shown also that during the transient phase the height of the specimen increases linearly with time

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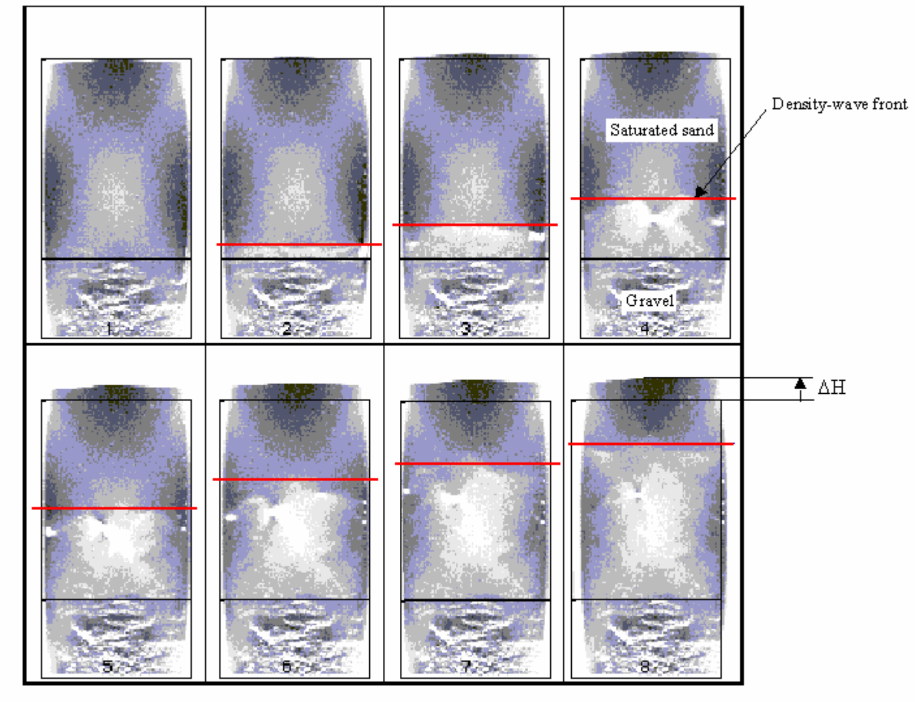
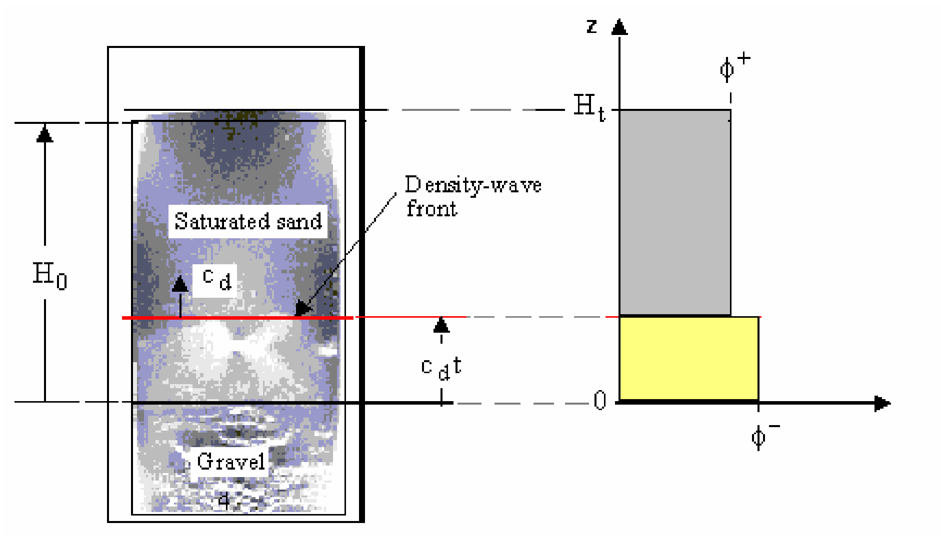
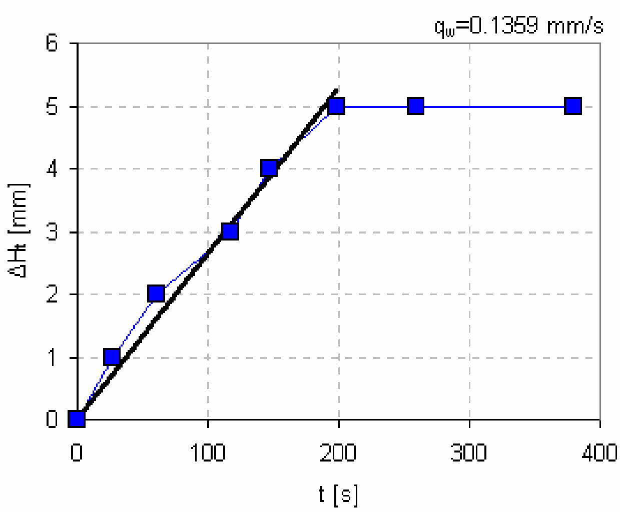
[10] For $\phi_0 = 0.34$, $G_s = \rho_s / \rho_w = 2.65$ we get $J_c = 1.089$.

$$H_t \approx H_0 + \dot{H}_t t$$







This allows us to assume that the grain velocity in front of the wave is approximately equal to the rate of height increase, whereas behind the front we assume that the sand is at rest,

$$v_s^+ = \dot{H}_t \quad , \quad v_s^- = 0$$

These assumptions allow us to determine the propagation speed of the porosity shock-wave by applying the Rankine-Hug?niot condition, eq. (3.38),

$$c_d = \frac{[[q^{(1)}]]}{[[\rho_1]]}$$

where in the present setting

$$\rho_1 = (1-\phi)\rho_s \quad , \; q^{(1)} = \rho_1 v^{(1)}$$

are the partial density and the mass flux of the solid phase, respectively.

The jumps are computed as follows,

$$[[\rho_1]] = \rho_s(1-\phi^+) - \rho_s(1-\phi^+) = -\rho_s(\phi^+ - \phi^-) = -\rho_s[[\phi]]$$

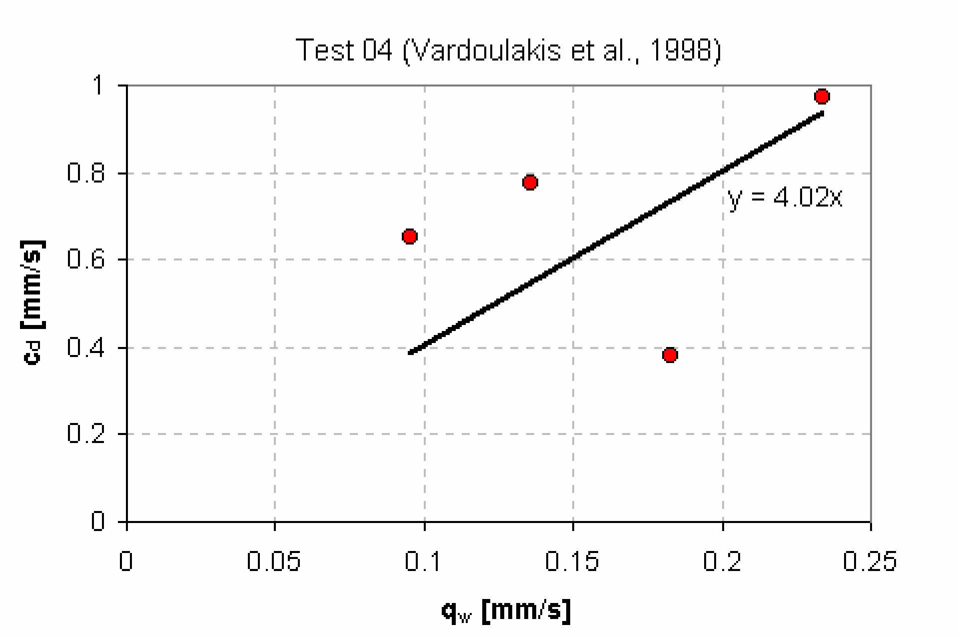
$$[[q^{(1)}]] = \rho_s(1-\phi^+)v_s^+ - \rho_s(1-\phi^-)v_s^- = \rho_s(1-\phi^+)\dot{H}_t$$

Thus the evaluation formula of the experimental measurements for determining the shock-wave speed is

$$c_d = \frac{1-\phi_{int}}{\phi_{fnl} - \phi_{int}}\dot{H}_t$$

The experiment has shown that $\dot{H}_t$, and with that c_d, are functions of the water flow-rate q_w,

$$c_d \approx \lambda q_w \quad , \quad \lambda \approx 4. \tag{3.39}$$



We remark finally that in porous media the flow rate is function of the applied hydraulic gradient. As we will see for example in a following chapter, the simplest law that relates the flow-rate to the pressure gradient is Darcy's law, $q = kJ$. The coefficient k is called the **permeability** of the medium and is fund to be a function of the porosity of the medium. This means here that the empirical relationship (3.9) implies that the porosity wave speed $c = C(\phi)$ is function of the porosity itself. Such a non-linear wave is called a **kinematic wave.**

4. SHALLOW-WATER WAVES

4.1 Balance of Linear Momentum

We consider again the motion of a 1D-continuum between two sections at the positions $x = a$ and $x = b$, with the lagrangian co-ordinates α and β, respectively. At time t these sections have the spatial coordinates

$$a = \chi^L(\alpha, t) = \hat{a}(t) \quad , \quad b = \chi^L(\beta, t) = \hat{b}(t)$$

In the previous chapters, the continuum has been equipped with the 'primitive' mechanical properties of mass density $\rho(x,t)$ and velocity $v(x,t)$. Thus a 'particle' or 'material point' of the 1D-continuum has the mass $dm = \rho\, dx$, the velocity v and accordingly it possesses the **linear momentum**[1],

$$dI = dm\, v = \rho v\, dx$$

This gives rise to the definition of the 'density' of linear momentum

$$i = \rho v$$

The total momentum of the material contained in the considered control volume between the two material sections is,

[1] The *'linear'* momentum of a particle is juxtaposed from its *'angular'* momentum, $d\Theta = \vec{r} \times (\rho \vec{v}) dV$, with respect to the origin of the position vector $\vec{r}$. Obviously the angular momentum enters in 2D- and 3D- considerations.

$$I(t)=\int_a^b i(x,t)\,dx$$

Balance of linear momentum requires that: The rate of change of total linear momentum in the considered control volume is equal to the total force acting on the mass in that volume[2],

$$\frac{dI}{dt}=F \qquad (4.1)$$

The forces that may act on the considered continuum are of two types:

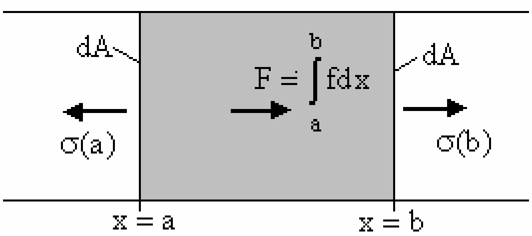
1. **Volume forces** fdx, which act on all material points in the control volume. In this category belongs the self-weight of a material body in the gravitational field. However, as we will see in a chapter 5, we may introduce other body forces as well, which are not field-forces. For example we may model the action of the interstitial fluid on the particles of a soil by introducing the so-called *seepage force*, which is a continuum body-force, that accounts for the friction of flowing fluid particles along grain boundaries.

2. **Boundary forces** dN, which appear at the boundary sections at the position $x=a$ and $x=b$, respectively. Boundary forces give rise to the definition of *stress*, as a surface density of force. Let dA be the cross-sectional area at a given position, then

$$dN=\sigma dA \qquad \text{or:} \qquad \sigma=\frac{dN}{dA} \quad , \quad [\sigma]=\frac{F}{L^2}$$

Accordingly the total force per unit cross-sectional area that is acting on the material between the two sections at $x=a$ and $x=b$ is

$$F=\int_a^b fdx+\sigma(b)-\sigma(a) \qquad (4.2)$$



[2] Lazare Carnot (1753-1823) was the first to connect in Solid Mechanics the concepts of change of momentum of body with mass m that moves with the velocity $\vec{v}$ to the impulse of the force $\vec{F}$ that acts on the body in the time interval dt, $d(m\vec{v})=\vec{F}dt$.

The rate of momentum in the considered control volume is computed here in a similar way as we did in the evaluation of the mass balance law. By using Reynold's transport theorem we get,

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\hat{a}(t)}^{\hat{b}(t)} i(x,t)dx = \int_a^b \frac{\partial}{\partial t} i(x,t)\,dx + v(b,t)\,i(b,t) - v(a,t)\,i(a,t) \tag{4.3}$$

With equations (4.2) and (4.3) the balance law of linear momentum (4.1) takes then the following form,

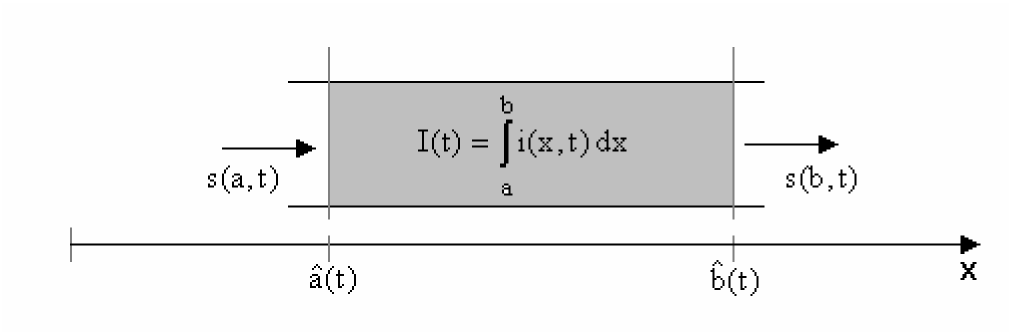
$$\int_a^b \frac{\partial i}{\partial t} dx + s\Big|_{x=a}^{x=b} = \int_a^b f dx + \sigma\Big|_{x=a}^{x=b} \tag{4.4}$$

where

$$\underline{s = i\,v = \rho\, v^2}$$

is the **momentum flux.**

This means that the material points are transporting momentum due to their mass and their velocity. This momentum is considered to flow into the control volume at section $x = a$ and flow out of it at section $x = b$.



Accordingly the balance law of linear momentum, eq. (4.4), is interpreted as follows: The total rate of change of momentum in the control volume is due to the action of boundary forces and due to the net <u>influx</u> of momentum

$$\int_a^b \frac{\partial i}{\partial t} dx = \int_a^b f dx + \sigma\Big|_{x=a}^{x=b} - s\Big|_{x=a}^{x=b}$$

Above momentum balance equation (4.4) can written also as

$$\int_a^b \left\{\frac{\partial i}{\partial t} + \frac{\partial s}{\partial x}\right\} dx = \int_a^b \left(f + \frac{\partial \sigma}{\partial x}\right) dx \tag{4.5}$$

The expression on the l.h.s. of eq. (4.5) is simplified significantly, if we assume the validity of the mass conservation law, since

$$\frac{\partial i}{\partial t} + \frac{\partial s}{\partial x} = \frac{\partial \rho}{\partial t} v + \rho \frac{\partial v}{\partial t} + \frac{\partial (\rho v)}{\partial x} v + \rho v \frac{\partial v}{\partial x}$$

$$= v\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right\} + \rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right)$$

Indeed, if mass is conserved, then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

and the first term inside the bracket vanishes. With the remark that

$$a = \dot{v} = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

is the **acceleration** of the material point, we get finally the following global from of the so-called *dynamical equation*,

$$\int_a^b \rho \dot{v} dx = \int_a^b \left(f + \frac{\partial \sigma}{\partial x} \right) dx \qquad (4.6)$$

The local form of the balance law of linear momentum is obtained under suitable assumptions concerning the continuity of the various involved fields. Thus from eq. (4.5) we get the following localization of the momentum balance law

$$\frac{\partial i}{\partial t} + \frac{\partial s}{\partial x} = f + \frac{\partial \sigma}{\partial x}$$

We remark that this equation has the general form of a **'conservation law'**[3],

$$\frac{\partial}{\partial t}(\textit{density}) + \frac{\partial}{\partial x}(\textit{flux}) = (\textit{generation})$$

In case where mass is conserved, from eq. (4.6) we get finally the well-known dynamical equation

$$\boxed{\rho \dot{v} = f + \frac{\partial \sigma}{\partial x}} \qquad (4.7)$$

[3] cf. eq. (3.5.bis)

Finally we remark that according to *D' Alembert's principle* we may identify the quantity $(-\rho\,\dot{v})$ as an *inertial 'force'*[4], by writing the dynamic equation as a static one

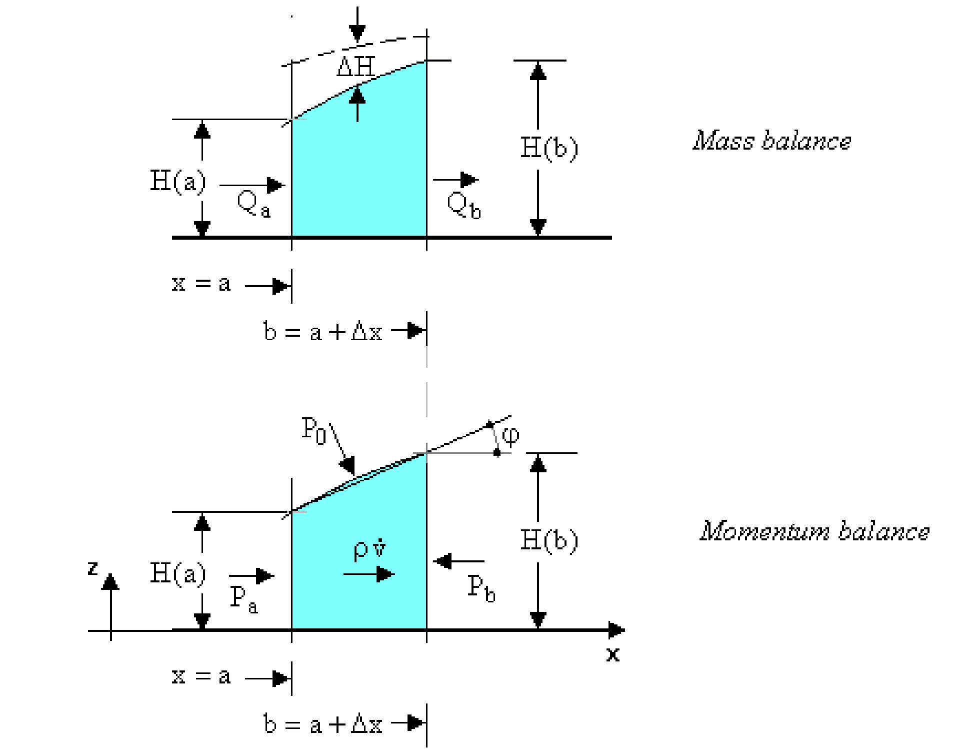
$$\frac{\partial \sigma}{\partial x} + \{f - \rho \dot{v}\} = 0$$

4.2 Theory of Shallow Water Waves[5]

As we did in section 3.1.1 we consider here an essentially one-dimensional motion of an incompressible fluid. In that case balance of mass has led to the following storage equation,

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial x}(Hv) \tag{3.5}$$

where H is the elevation of the free water surface from the bottom of the considered water sheet and v is the (height averaged) fluid velocity.



In order to evaluate the balance of linear momentum equation we assume that water is an inviscid fluid, i.e. a fluid that is capable to carry a pressure field,

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[4] Inertial 'forces' are pseudo-forces, since they do not obey to Newton's 3rd law, which demands that forces appear in pairs ("actio-reactio"). For example if the body force **f** is a gravity force, then its counterpart **- f** acts on the attracting mass, namely on the Earth.

[5] De Barr Saint-Venant, A.J.C. (1850). Mémoire sur des formules nouvelles pour la solution des problèmes ralitfs aux eaux courantes. C.R. Acad. Sc., Paris, 31, 283. De Barr Saint-Venant, A.J.C. (1871). Théorie du movement non-permanent des eaux avec applications aux crues des rivieres et a l' introduction des marées dans leurs lit. C.R. Acad. Sc., Paris, 73, 147-154. H. Lamb, *Hydrodynamics*, Cambridge University Press, (1879, rep. 1995).

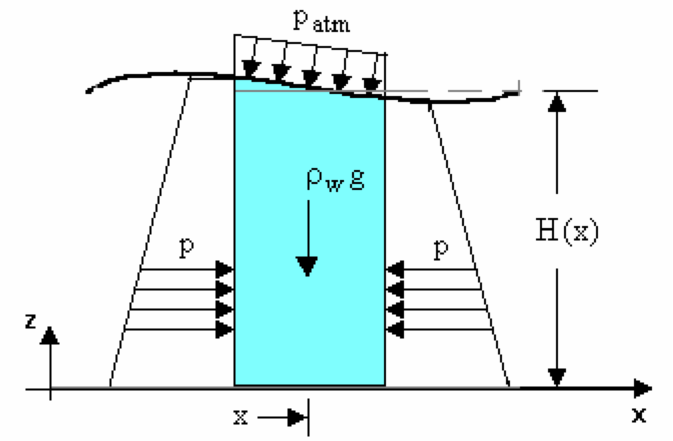
$\sigma = -p(x,z,t)$, and no shear stresses. In addition we assume that the pressure field is practically hydrostatic,

$$p = p_{atm} + \rho_w g\left(H(x,t) - z\right) \tag{4.8}$$

In this expression z is the vertical coordinate measuring positive upwards from the bottom of the water sheet. p_{atm} is the atmospheric pressure, $\rho_w = \gamma_w / g$ is the density of water, $\gamma_w = 10.\,kN/m^3$ is the unit weight of water and $g = 9.81 m/s^2$ is the gravity acceleration.

Exercise

Prove that equation (4.8) follows from the assumption that in the considered setting vertical particle accelerations are negligible ($|\dot{v}_z| \to 0$).



The global dynamic equation for a slice of water between the cross-sections at the positions $a = x$ and $b = x + dx$ reads as follows,

$$P_a - P_b + P_0 \sin\varphi = \rho \dot{v} H dx$$

where

$$P(x,t) = \int_0^{H(x,t)} p(x,z,t)dz = p_{atm}H(x,t) + \frac{1}{2}\rho_w g H(x,t)^2 \quad \Rightarrow$$

$$P(a,t) = p_{atm}H(a,t) + \frac{1}{2}\rho_w g H(a,t)^2$$

$$P(b,t) = P(a,t) + \left.\frac{\partial P}{\partial x}\right|_{x=a} dx = P(a,t) + p_{atm}\left.\frac{\partial H}{\partial x}\right|_{x=a} dx + \rho_w g H(a,t)\left.\frac{\partial H}{\partial x}\right|_{x=a} dx$$

and

$$P_0 \sin\varphi = p_{atm} dx \sin\varphi \approx p_{atm} dx \tan\varphi = p_{atm} \left.\frac{\partial H}{\partial x}\right|_{x=a} dx$$

This leads to the following local form of the dynamical equation in horizontal direction,

$$-\rho_w g H \frac{\partial H}{\partial x} = \rho_w H \dot{v}$$

By utilizing the definition of the acceleration as the material time derivative of the velocity we get the following dynamical equation

$$-g\frac{\partial H}{\partial x} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \qquad (4.9)$$

Accordingly the set of governing equations consists is the storage equation (3.4) and the dynamical equation (4.9),

$$\frac{\partial H}{\partial t} + v\frac{\partial H}{\partial x} + H\frac{\partial v}{\partial x} = 0 \qquad (3.4)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial H}{\partial x} = 0 \qquad (4.9)$$

As we will see below these equations describe the propagation of waves in shallow water-sheets. These waves are called also **'gravity waves'** since gravity is the only force that drives the particles back to their original position.

4.3 Linear Theory of 'Tidal' Waves

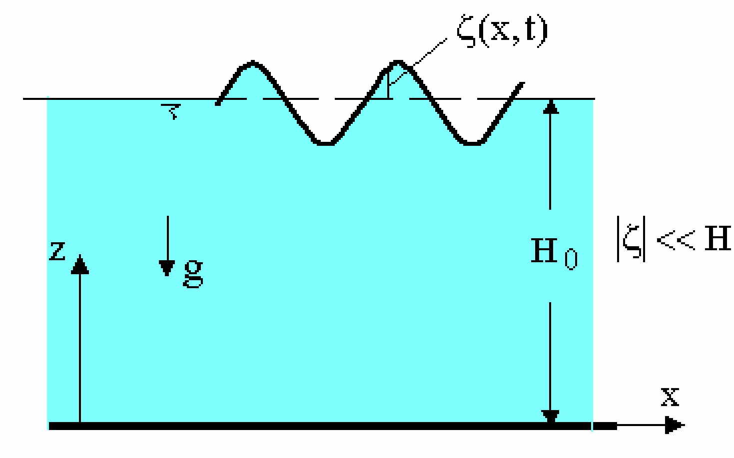
The partial differential equations (3.5) and (4.9) are non-linear due to the product terms $v(\partial H/\partial x)$ etc. In order to develop a linear theory of waves in shallow water sheets we assume that the free surface elevations can be written in the form of a 'small' *perturbation* out of its position at rest

$$H = H_0 + \zeta(x,t) \quad , \quad |\zeta| << H_0 \qquad (4.10)$$

Under the assumption that the amplitude ζ of the wave as well as that all gradients of the involved fields are also 'small' quantities, the governing equations (3.4) and (4.9) can be linearized as follows: We introduce into equations (3.5) and (4.9) the expression (4.10) for the free-surface elevation and we neglect products of 'small' quantities,

$$\frac{\partial\zeta}{\partial t}+v\frac{\partial\zeta}{\partial x}+(H_0+\zeta)\frac{\partial v}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial\zeta}{\partial t}+H_0\frac{\partial v}{\partial x}=0 \qquad \text{(a)}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial\zeta}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial v}{\partial t}+g\frac{\partial\zeta}{\partial x}=0 \qquad \text{(b)}$$



From the resulting equations we may eliminate the velocity,

$$\left.\begin{aligned}\text{(a)}:&\quad \frac{\partial^2\zeta}{\partial t^2}=-H_0\frac{\partial^2 v}{\partial t\partial x}\\ \text{(b)}:&\quad \frac{\partial^2\zeta}{\partial x^2}=-g\frac{\partial^2 v}{\partial x\partial t}\end{aligned}\right\}\Rightarrow\quad \underline{\frac{\partial^2\zeta}{\partial t^2}=c_0^2\frac{\partial^2\zeta}{\partial x^2}}\quad,\quad c_0=\sqrt{gH_0}$$

The resulting partial differential equation is called the **wave equation**, because, as we will see below, it admits wave-like solutions. Indeed the wave equation can be written in operational form as follows,

$$\left(\frac{\partial}{\partial t}+c_0\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial}{\partial t}-c_0\frac{\partial}{\partial x}\right)\zeta(x,t)=0$$

This means that the original equation splits into two equations, the so-called

- *'foreword' wave equation*: $\dfrac{\partial\zeta}{\partial t}-c_0\dfrac{\partial\zeta}{\partial x}=0$ (4.11a)

- *'backward' wave equation*: $\dfrac{\partial\zeta}{\partial t}+c_0\dfrac{\partial\zeta}{\partial x}=0$ (4.11b)

The general solution of eq. (4.11a) is a 'foreword' moving wave-form,

$\zeta = f(x - c_0 t)$

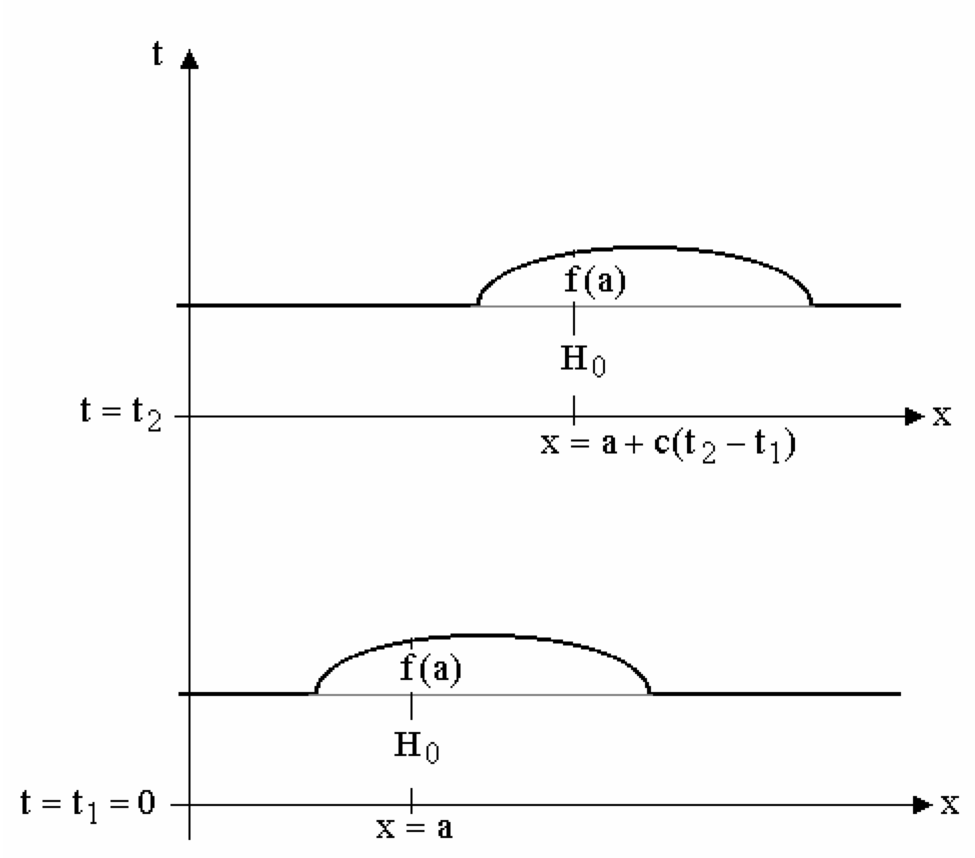
and the general solution of eq. (4.11b) is of course a 'backward' moving wave-form,

$\zeta = f(x + c_0 t)$

The wave-form solution

$\zeta = f(x \pm c_0 t)$

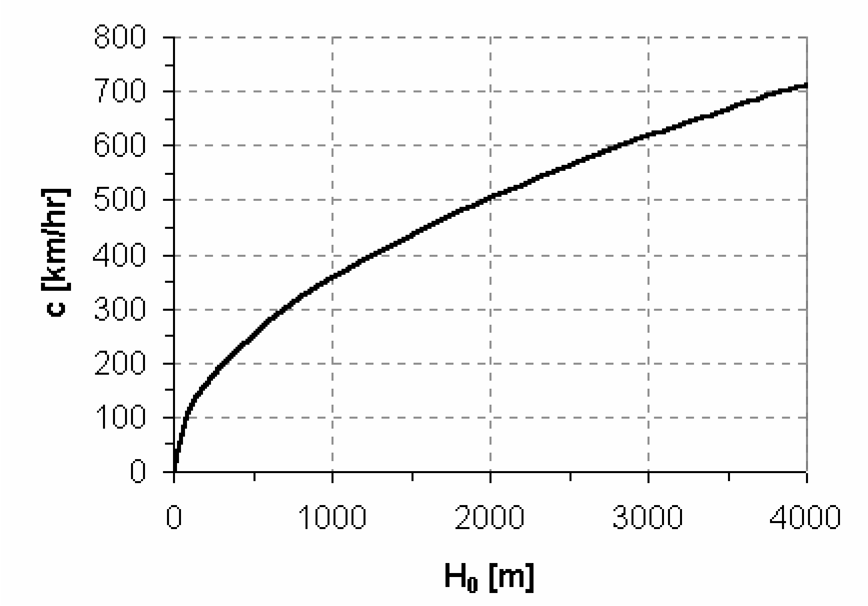
is known as the *D' Alembert solution*[6] *.*



This means that surface waves is shallow waters with small amplitude propagate with constant speed that depends only on the at-rest thickness of the considered water sheet

$c_0 = \sqrt{gH_0}$

[6] Jean le Rond D'Alembert 1717-1783.



4.5 The Tsunami[7]

A special form of extremely dangerous gravity waves in 'shallow waters' is the so-called **tsunamis**. For example in the Pacific Ocean, with an average depth of 4 km, a tsunami will move with a speed of about 700 km/hr. In the Aegean Sea, with an average depth of about 500m, the Thera tsunami of 1400 BC must have moved with a speed of about 225 km/hr and it must have taken about 30 min to reach and destroy the Minoan civilization in the north shore of Crete[8].

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[7] Japanese: tsu = harbor, nami = wave." ….Tsunamis are actually trains of up to ten or more waves. In contrast to wind-generated waves, in which water is momentarily displaced vertically, tsunami waves actually transport water forwards and backwards. And, although the wave is shortened from its deep-water wavelength, as it approaches shore, it still extends several kilometers crest to crest, says wave mechanics expert Synolakis. Photographs of tsunamis coming in sometimes look like an approaching giant stair-step that stretches as far as the eye can see. Eyewitnesses often speak of water that just kept on coming. Though tsunamis are usually generated by underwater earthquakes, they may be caused by any massive disturbance of the water column, either from above or below. (Although older accounts often refer to tsunamis as "tidal waves," one natural phenomenon that doesn't cause tsunamis is the tides.) They may, for instance, be caused by large volcanic eruptions, like the infamous August 27, 1883, eruption of Krakatau in the East Indies. The sudden collapse of the 2,000-meter high volcano triggered a powerful tsunami that killed over 36,000, and left no trace of 165 coastal villages. In rare instances, the impact of celestial objects such as meteors can cause tsunamis. Two types of landslides can generate tsunamis that, although localized, can be immense. Subaerial slides begin on land, but their debris can impact the water. Submarine slides begin and end completely under water. The largest tsunami-like wave on record occurred on July 10, 1958, in Lituya Bay, Alaska, following a local earthquake, which loosened 90 million tons of rock at one end of the Bay. The landslide caused a surge that rose to heights of 500 meters on the opposite shore. Submarine landslides not associated with earthquakes can be particularly insidious, generating tsunamis that arrive with no warning. Such slides could occur, for example, in the steep-walled submarine canyons common off the California coast…." http://www.calacademy.org/calwild/spring99/tsunamis.htm

[8] K. Minoura et al. (2000). Discovery of Minoan tsunami deposits. Geology, **28**, 59-62.

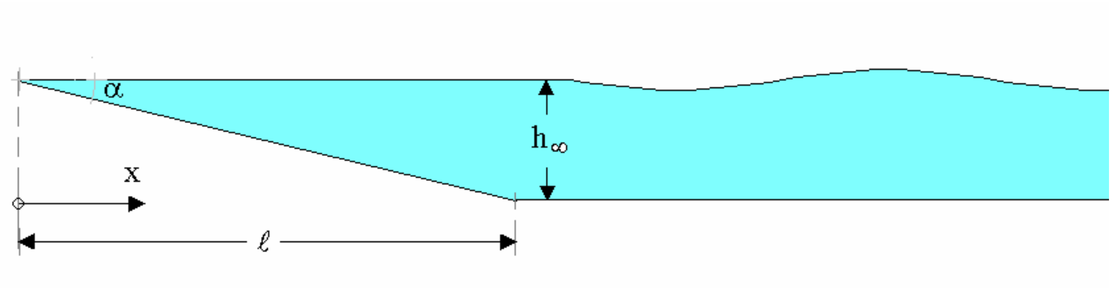
For the mathematical modeling of the phenomenon of wave amplification near to the shore we remark that a slightly more general for of the governing equations (a) and (b) is the following:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left(h_0(x)\, v\right) = 0 \qquad (4.12a)$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0 \qquad (4.12b)$$

where we have set that the thickness of the water at rest is a function of x, $H_0 = h_0(x)$. Now, we assume that the see-bottom is gently inclined upwards as we approach the shore. For simplicity we deal here with some-kind of mean slope angle, $\alpha \approx \tan\alpha = h_\infty / \ell$, and with that we set

$$h_0(x) = \begin{cases} \alpha x & (0 \le x \le \ell) \\ h_\infty & (\ell \le x) \end{cases}$$



In that case elimination of the velocity from eqs. (4.12) yields

$$\frac{\partial^2 \zeta}{\partial t^2} - g\alpha x\frac{\partial^2 \zeta}{\partial x^2} - g\alpha\frac{\partial \zeta}{\partial x} = 0 \quad (0 \le x \le \ell) \qquad (4.13)$$

We assume that a wave has the form $\zeta = H(x)\cos(\omega t + \varepsilon) \quad (0 \le x \le \ell)$, with the boundary condition $H(\ell) = a$. In that case eq. (4.13) yields,

$$x\frac{d^2 H}{dx^2} + \frac{dH}{dx} + \frac{\omega^2}{\alpha g}H = 0$$

We set

$$x = \frac{\alpha g}{\omega^2}\left(\frac{\xi}{2}\right)^2$$

and above equation becomes a Bessel differential equation

$$\frac{d^2H}{d\xi^2} + \frac{1}{\xi}\frac{dH}{d\xi} + H = 0$$

The general solution of the Bessel equation is

$$H = AJ_0(\xi) + BY_0(\xi)$$

where ? and ? are integration constants and $J_0(\xi), Y_0(\xi)$ are the Bessel functions of zero order and of the 1st and 2nd kind respectively[9],

$$J_0(\xi) = 1 - \frac{\frac{1}{4}\xi^2}{(1!)^2} + \frac{\left(\frac{1}{4}\xi^2\right)^2}{(2!)^2} - \frac{\left(\frac{1}{4}\xi^2\right)^3}{(3!)^2} + \cdots$$

$$Y_0(\xi) = \frac{2}{\pi}\left(\ln\left(\frac{\xi}{2}\right) + \gamma\right)J_0(\xi) + \frac{2}{\pi}\left\{\frac{\left(\frac{1}{4}\xi^2\right)}{(1!)^2} - \left(1+\frac{1}{2}\right)\frac{\left(\frac{1}{4}\xi^2\right)^2}{(2!)^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\left(\frac{1}{4}\xi^2\right)^3}{(3!)^2} - \cdots\right\}$$

$$\gamma = 0.5772156649\ldots$$

If we requiere that the solution is bounded at $\xi = 0$ ($|H(0) < \infty|$), then the Neumann-Weber logarithmic solution $Y_0(\xi)$ is excluded $(B = 0)$, and

$$H = AJ_0(\xi) \quad , \quad \xi = 2\omega\sqrt{\frac{x}{\alpha g}}$$

With the consideration of the aforementioned boundary condition this finally yields to the following solution

$$\zeta = a\frac{J_0\left(2\omega\sqrt{\frac{x}{\alpha g}}\right)}{J_0\left(2\omega\sqrt{\frac{\ell}{\alpha g}}\right)}\cos(\omega t + \varepsilon) \qquad (4.14)$$

As can be easily seen this solution implies an amplification that goes together with decrease in wave length as the wave approaches the shore.

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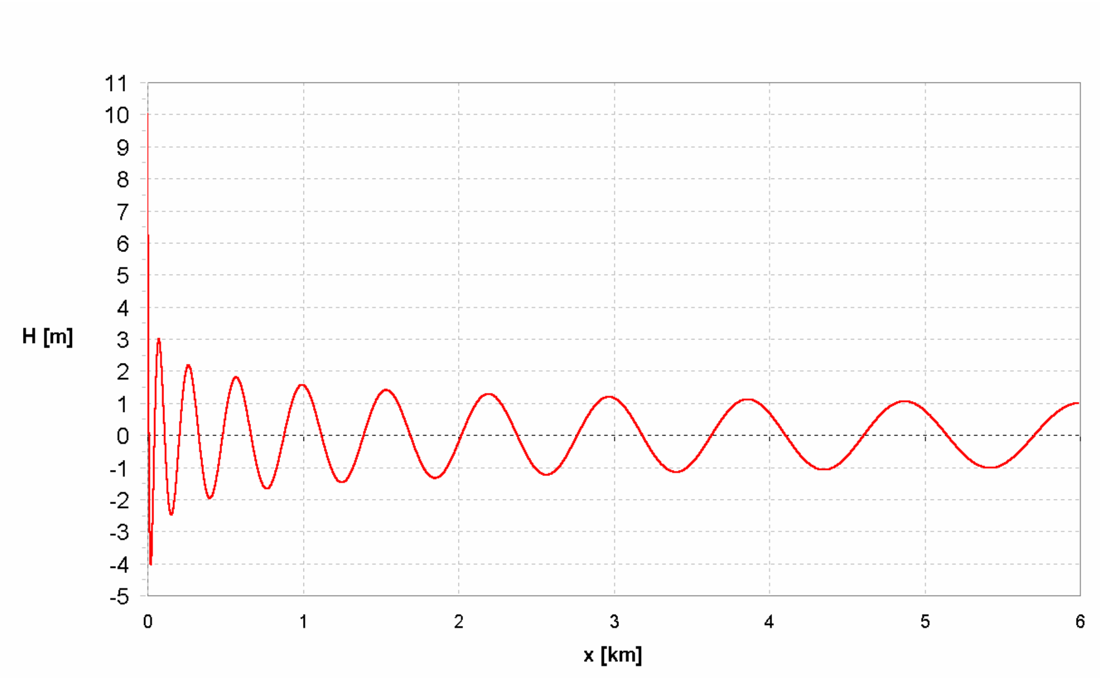
[9] cf. Bowman, F. *Introduction to Bessel Functions*. Dover, 1958.



Example: $\ell = 6.0\text{km}$, $\alpha = 10.^{o}$, $\omega = 0.54\ \text{s}^{-1}$, $h_\infty = 1058.\text{m}$,
$c_\infty = \sqrt{g h_\infty} = 367.\ \text{km/hr}$.

4.5 Non-linear Water Waves

4.5.1 Kinematic waves

We consider the fully non-linear balance equations (3.4) and (4.9) and assume that there is a **kinematic constraint** between the (height averaged) particle velocity in horizontal direction and the height of the free surface,

$$\underline{v = V(H)} \tag{4.14}$$

Under this assumption equations (3.5) and (4.9) become

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + H V'\frac{\partial H}{\partial x} = 0$$

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + \frac{g}{V'}\frac{\partial H}{\partial x} = 0$$

where

$$V' = \frac{dV}{dH}$$

If we require that both equations describe the same phenomenon, then we obtain a differential equation for the kinematic constrain function between velocity and height,

$$V' = \pm\sqrt{\frac{g}{H}} \quad \Rightarrow \quad V = \pm\, 2\sqrt{g\,h} + \text{const.}$$

The integration constant is determined from the condition, that at the equilibrium elevation H_0 the velocity is vanishing,

$$V(H_0) = 0$$

This is leading to the following kinematic constraint

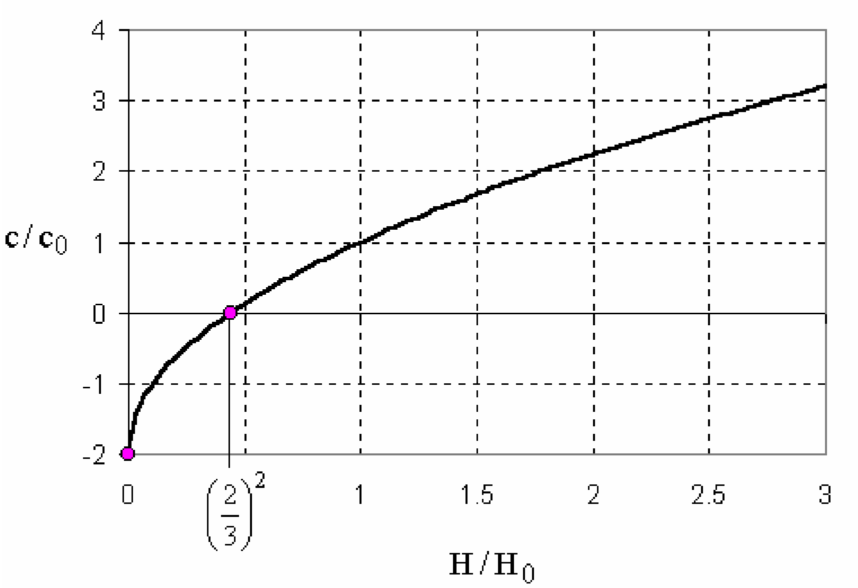
$$v = V(H) = \pm(2\sqrt{g\,H} - 2\sqrt{g\,H_0})$$

and the problem is reduced to the following partial differential equation

$$\frac{\partial H}{\partial t} \pm c(H)\frac{\partial H}{\partial x} = 0 \qquad (4.15)$$

with

$$c = \left(3\sqrt{\frac{H}{H_0}} - 2\right)c_0 \quad , \quad c_0 = \sqrt{gH_0} \qquad (4.16)$$



As we will see in the following the partial differential equation (4.15) is again describing the propagation of wave-forms. Due to the fact that eq. (4.15) resulted from the assumption of the kinematic constraint (4.14), the corresponding waves are called **kinematic waves**. Since the wave speed c is not constant but a function of the actual height of the wave, $c = C(H)$, eq. (4.16), the corresponding waves are called also **non-linear waves**. For $c > 0$, eq. (4.15) with the (+)-sign describes **foreword-waves** that move to the right $(\rightarrow)$ and with the (-)-sign it describes **backward-waves** that move to the left $(\leftarrow)$. In the special case, where the difference, $|H - H_0| \to 0 \quad \Rightarrow \quad c \to c_0$, we recover the case of linear water-waves.

4.5.2 The method of characteristics

We discuss here solutions of the 'foreword' equation,

$$\frac{\partial H}{\partial t} + c(H)\frac{\partial H}{\partial x} = 0 \tag{4.17}$$

We assume that at time $t = 0$ the *initial condition* for the height H is known

$$H(x,0) = h(x)$$

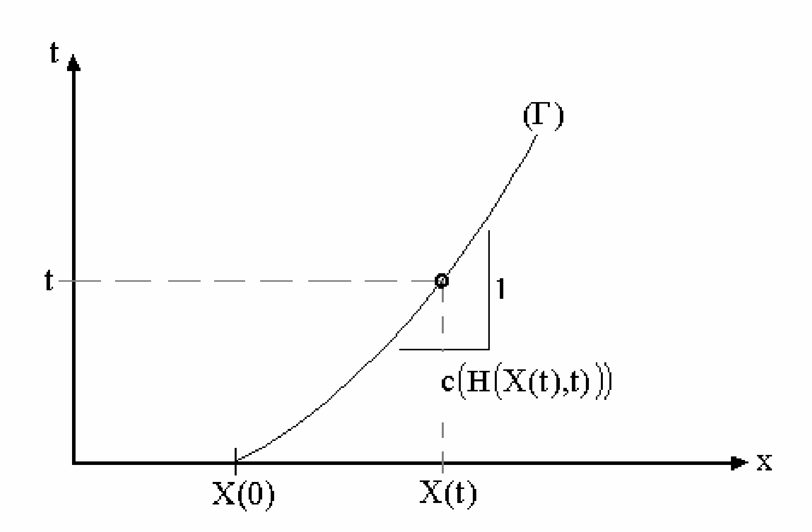
and we want to compute the space-time evolution of $H(x,t)$ and of $v = V(H)$.

We consider in the plane $O(x,t)$ of events a curve (Γ), which is described by the equation,

$(\Gamma): \quad x = X(t)$

such that

$$\frac{dX}{dt} = c\big(H(X(t),t)\big)$$



Along the curve (Γ) the height H is a function only of time t

$$H = H\big(X(t), t\big) = \hat{H}(t)$$

Thus along the curve (Γ) we get,

$$\frac{d\hat{H}}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\frac{dX}{dt}$$

Due to the definition of the curve (Γ) and due to the governing equation (4.13f) we get that along the curve (Γ) the function H remains constant,

$$\frac{d\hat{H}}{dt} = \frac{\partial H}{\partial t} + c\,\frac{\partial H}{\partial x} = 0 \quad \Rightarrow \quad \hat{H} = \text{const.}$$

Moreover we observe along such a curve (Γ) both,

$$c(\hat{H}) = \text{const.} \quad \Rightarrow \qquad \frac{dX}{dt} = \text{const.}$$

This means that the curve (Γ) is a **straight line** in the plane $O(x,t)$ of events. Such a curve (Γ) s called a **characteristic line** or simply a **characteristic** of equation (4.17)[10].

Following these observations we may describe here briefly a general method of solution of the considered initial-value problem, utilizing the concept of characteristics. The corresponding solution method is known as the

'**Method of Characteristics'**:

1. The x-axis is discretised.

2. We evaluate the initial condition for the height at the discrete points on the x-axis,

$$H(x,0) = h(x)$$

3. At any individual point $x = s$, along the x-axis ($t = 0$), we compute the propagation velocity

$$c_0(s) = c\big(h(s)\big) = 3\sqrt{gh(s)} - 2\sqrt{gH_0}$$

4. In the plane of events $O(x,t)$ and through the point on the x-axis, with co-ordinates $(s,0)$ we draw the straight-line characteristic

$$(\Gamma): \quad x = s + c_0(s)\,t$$

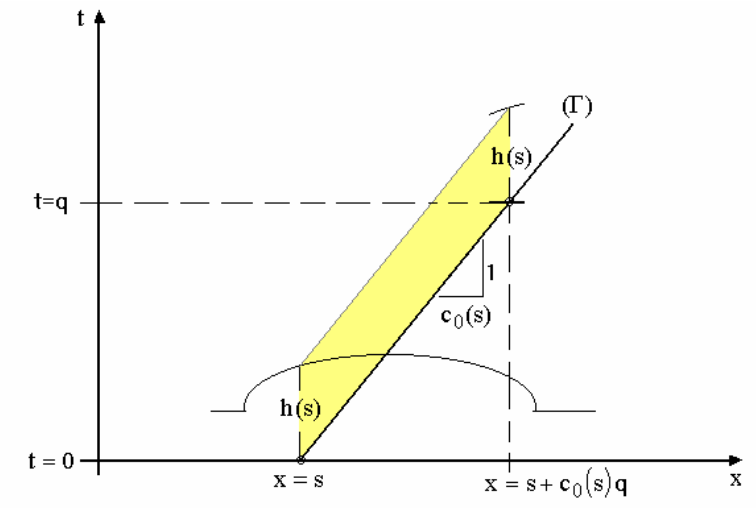
5. Along this characteristic straight line the information concerning the height and the velocity is transferred constant,

[10] In the Applied Mathematics literature a partial differential equation which has real characteristics is classified as a **hyperbolic equation**.

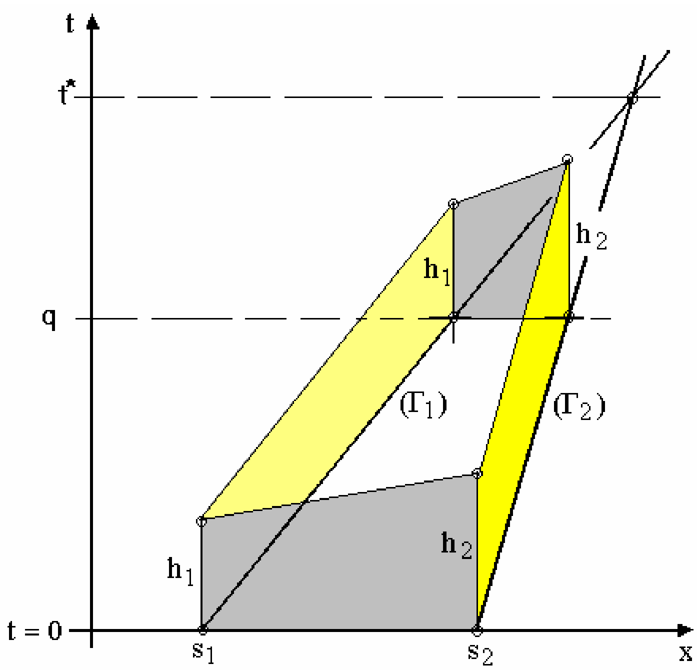
$$(\Gamma): \quad H(s + c_0(s)t, t) = H(s,0) = h(s)$$

$$(\Gamma): \quad V(s + c_0(s)t, t) = 2\sqrt{g\,h(s)} - 2\sqrt{g\,H_0}$$

6. If we want to evaluate the free-surface and velocity profiles at a given time $t = q > 0$, we intersect the characteristics with the line, $t = q$, and plot over the intersection points the values for H and $V(H)$, which they carry.

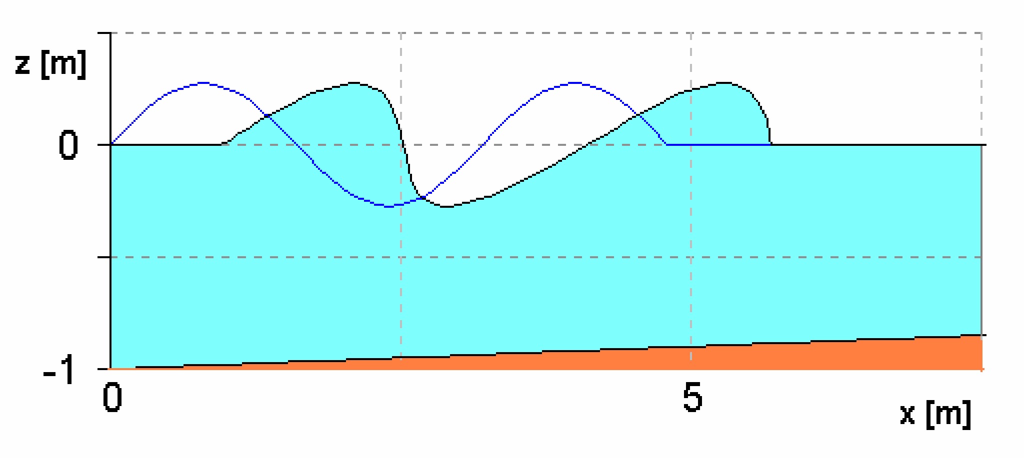


We observe that the wave propagation velocity is increasing with increasing height of the wave. This means that an ascending wave profile will get steeper as time goes on. The corresponding characteristic lines converge, and at some critical time mark $t = t^*$ they intersect. This means that at that event we have two values for the height (and the velocity) of the wave. This condition describes the beginning of a **'breaking' wave**. Past this point the present model is not valid and higher-order approximations are proposed in the pertaining literature.



4.5.3 Exercise

The following sketch is illustrating the theoretical prediction of the evolution of a sinusoidal $1\frac{1}{2}$-wave close to shore, moving from left to right. The sea-bottom is sloping upwards to the right, as one approaches the shore line.



Given are:

1. The slope of the sea bottom: $+1.5\%$.
2. The height at rest of the water sheet in the position $x = 0$: $H_0 = 2.m$,
3. The amplitude of the sinusoidal wave: 0.25 m
4. The Wavelength: 2 m.

Draw the free surface at different times and find the time t^* at which this wave will first start to break.

5 FLOW IN POROUS MEDIA

5.1 Darcy's Law

We consider a real (viscous) fluid, which is flowing in a straight pipe of constant circular cross-section. Let R be the inner radius of the pipe. The normal section of the fluid body has a perimeter $U = 2\pi R$ and an area $A = \pi R^2$. We consider the fluid between the cross-sections at the positions $x = a$ and $x = b$. Let $L = b - a$ be the length of the considered segment.

If $Q(x,t)$ denotes the discharge of fluid at any given cross-section, **continuity** of flow requires that

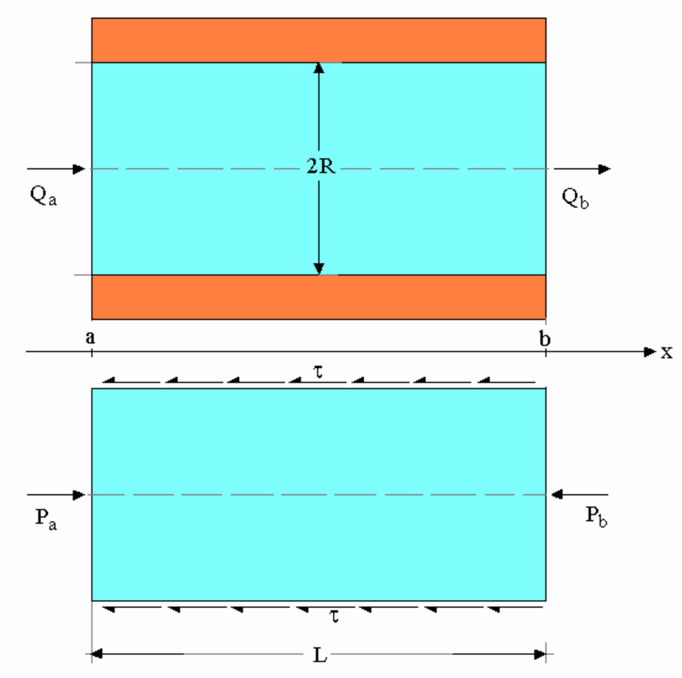
$$Q(a,t) = Q(b,t) = Q(x,t) = \hat{Q}(t)$$

The cross-sectional average of the flow velocity is computed as

$$v = \frac{Q}{A} = V(t) \tag{5.1}$$

Let $p(x,t)$ be the fluid pressure at any cross-section. The forces acting on the fluid body at the considered normal sections are

$$P_a = p(a,t)A \quad \text{and} \quad P_b = p(b,t)A$$



At the interface between flowing fluid and the inner pipe-wall, shear stresses τ develop, which are due to the fluid's internal friction. According to the Poiseuille solution[1], the interface friction shear-stress in circular pipes is proportional to the mean cross-sectional flow velocity,

$$\tau = \frac{4\mu_f}{R} v$$

In this expression μ_f is the **fluid viscosity** with dimensions $[\mu_f] = F\,T\,L^{-2}$.

Within a 1D-analysis the interfacial friction is replaced by a fictitious 'body' force f_ℓ (ℓ: lineal) that accounts for the total friction force acting on the flowing fluid per unit length of the pipe

$$f_\ell = \frac{\tau U L}{L} = \tau U = 8\pi\mu_f\, v \qquad (5.2)$$

This body force is a **non-conservative** force; it is a **follower-type** body force, since it follows the shape of the axis of the pipe. This body force has the dimensions of force per unit length in the axial direction,

$$[f_\ell] = FL^{-2}TLT^{-1} = FL^{-1}$$

[1] Jean L.M. Poiseuille (1799-1869) was a physician mainly interested in the physiology of blood flow. In contemporary terms he should be seen as pioneer in bio-engineering. So he studied the flow of liquids in small-diameter pipes of about 0.01mm (L.M. Poiseuille, *Recherches experimentales sur le mouvement des liquides*, Memoires, t. IX, p.433, Inst. Acad. Royale des Sciences.)

Let ρ_f be the fluid density. The **dynamical equation** for an infinitesimal segment $a = x$, $b = x + dx$ reads

$$P(x,t) - P(x + dx,t) - f_\ell dx = \rho_f A\dot{v}$$

or

$$-p(x,t)A - \left(-p(x,t) - \frac{\partial p}{\partial x}dx\right)A - f_\ell dx = \rho_f A\dot{v}$$

or

$$f - \frac{\partial p}{\partial x} = \rho_f \dot{v} \tag{5.3}$$

where

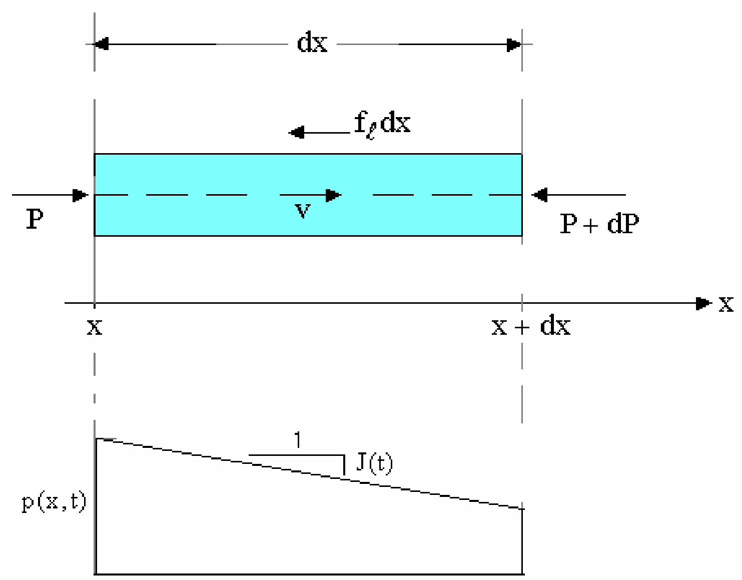
$$f = \frac{f_\ell}{A}$$

is the 'body' friction force f ($[f] = FL^{-3}$). From eqs. (5.2) and (5.4) we get that, according to Peuseuille's law, f is proportional to the mean flow-velocity

$$f = cv \tag{5.4}$$

and that the coefficient of viscous friction is proportional to the fluid viscosity and inversely proportional to the square of the radius of the pipe,

$$c = \pi\frac{\mu_f}{R^2} \tag{5.5}$$



From the basic balance laws (5.1) and (5.3) and the constitutive law (5.4) for the friction force, we obtain the following differential equation,

$$-\frac{\partial p}{\partial x} = \rho_f \frac{dv}{dt} + cv$$

or

$$-\frac{\partial p}{\partial x} = J(t) \quad \Rightarrow \quad p = -J(t)\,x + f(t)$$

where

$$J(t) = \rho_f \frac{dv}{dt} + cv$$

Thus at any instant the pressure distribution along the pipe is linear with slope $-J(t)$.

We consider now the special case, when between the two sections at $x = a$ and $x = b$ the pressure gradient is kept constant

$$p = p(a) - (x-a)J \quad \Rightarrow \quad J = -\frac{dp}{dx} = \frac{p(a) - p(b)}{L} = \text{const.}$$

In that case the mean cross-sectional velocity is given by the following linear differential equation,

$$\rho_f \frac{dv}{dt} + cv = J = \text{const.}$$

or

$$v = e^{-\int \frac{c}{\rho_f} dt} \left\{ \int \frac{J}{\rho_f} e^{\int \frac{c}{\rho_f} dt} dt + C \right\} \quad \Rightarrow$$

$$v = V(t) = \frac{J}{c} + Ce^{-(c/\rho_f)t}$$

The integration constant C is determined from the initial condition for the flow velocity

$$V(0) = \frac{J}{c} + C \quad \Rightarrow \quad v = \frac{J}{c} + \left(V(0) - \frac{J}{c} \right) \exp\left(-\frac{c}{\rho_f} t \right)$$

According to the above findings, the general solution for the flow velocity consists of two terms,

$$v = \bar{v} + \tilde{v}(t)$$

a) the **steady** solution (i.e. the time-independent solution)

$$\bar{v} = \frac{1}{c} J$$

b) The **transient** solution

$$\tilde{v} = \left(V(0) - \bar{v}\right) e^{-(c/\rho_f)t}$$

The transient solution fades exponentially with time, and the influence of the, generally unknown, initial condition for the flow velocity is rapidly erased. Thus, within a good approximation, the flow velocity obeys a **gradient law** of the form,

$$v \approx \bar{v} = \frac{1}{c} J \quad \Rightarrow \quad \underline{v \approx -\frac{1}{c}\frac{dp}{dx}} \tag{5.6}$$

If Δp the pressure drop from one end of the control volume to the other

$$\Delta p = p(b) - p(a) < 0$$

then the total discharge is given by the following expression,

$$\underline{Q = Av = -\left(\frac{A}{c}\right)\frac{\Delta p}{L}} \tag{5.7}$$

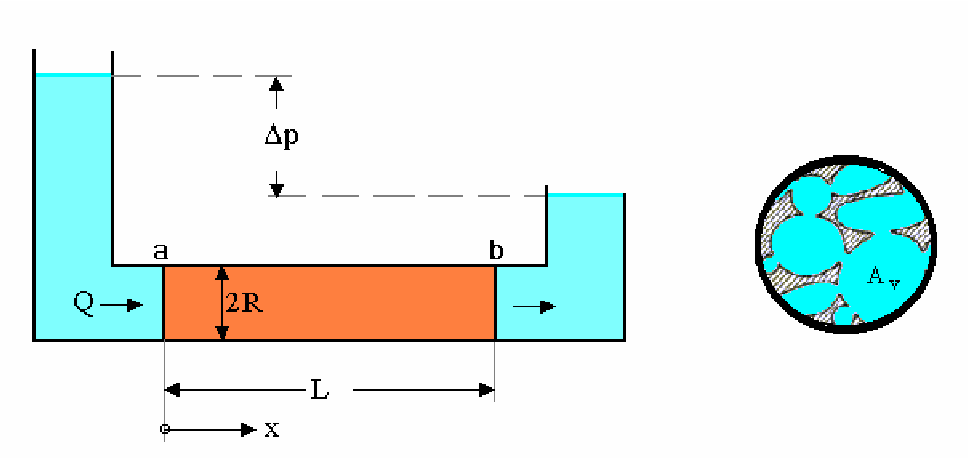
This relation is due to the French Engineer H. Darcy[2].

5.2 Flow in Porous Media

We consider the experimental set-up of the figure below: A porous material (e.g. a specimen of water-permeable soil) is flown axially with some fluid as it is placed inside a tube of inner radius R. The specimen has a length L and at its ends constant fluid-pressures are applied. The pressure head difference between the two ends of the specimen is ?h, corresponding to a pressure difference

$$\Delta p = -\rho_f g \Delta h$$

[2] H. Darcy, *Les fontaines publiques de la ville de Dijon*, 1856



If we focus our attention to sections of the specimen that are normal to the flow, then we can imagine that the wetted area A_v (v = void) corresponds to a pipe of irregular shape and we may apply Darcy's law, eq. (5.7) to this hypothetical pipe

$$Q = -\left(\frac{A_v}{c}\right)\frac{\Delta p}{L} \tag{5.8}$$

The wetted area is given as a fraction of the total area,

$$A_v = \phi_A A$$

where ϕ_A is the *surface porosity* of the medium.

For statistically isotropic random media we assume that the surface porosity is equal to the volumic porosity

$$\phi_A = \frac{A_v}{A} \approx \frac{V_v}{V} = \phi$$

and thus

$$A_v \approx \phi A$$

We introduce the so-called specific fluid discharge

$$q = \frac{Q}{A}$$

and observe that between q and the mean particle velocity of the interstitial fluid holds,

$$v = \frac{Q}{A_v} = \frac{Q}{A}\frac{A}{A_v} = \frac{q}{\phi} \quad \Rightarrow \quad q = \phi v$$

In that case Darcy's law, eq. (5.8) becomes

$$q = \frac{Q}{A} = -k_f \frac{\Delta p}{L}$$

where

$$k_f = \frac{\phi}{c}$$

is called the **permeability coefficient** of the porous medium with respect to the considered fluid (subscript f). From eq. (5.5) we learn that the permeability coefficient must be proportional to the square of the characteristic diameter D_c of the pores and inversely proportional to the fluid viscosity,

$$k_f = f(\phi)\frac{D_c^2}{\mu_f}$$

The normalized permeability coefficient with respect to the fluid viscosity

$$k = \frac{k_f}{\mu_f}$$

is called the **physical permeability coefficient** of the porous medium after Muskat[3]. The Muskat permeability has the dimension of square length. The physical permeability depends on factors that determine the geometric properties of the irregular pore-pipe

$$k = k(\phi, D_c^2 ...)$$

For many sands it turns out that the characteristic pore-size D_c is proportional to the 'hydraulically effective' grain diameter $D_{10\%}$, according to the well-known *Hazen rule*[3]

$$k \propto D_{10\%}^2$$

In the pertinent literature[4] one finds alternatively the so-called *Carman-Kozeny* equation for he physical permeability,

$$k = c' \frac{\phi^3}{(1-\phi)^2}\left(\frac{D_{50\%}}{6}\right)^2$$

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[3] D.W. Taylor, *Fundamentals of Soil Mechanics*, Wiley, 1948
[4] J.Bear, *Dynamics of Fluids in Porous Media*, Dover,1988

where $c' = 0.2$ is the so-called *Kozeny* constant reflecting the effect of pore-space tortuotity.

The experimentally verified law of Darcy may be generalized to the following constitutive relationship between fluid discharge and pressure gradient,

$$q = -\frac{1}{f}\frac{\partial p}{\partial x} \quad , \quad f = \frac{\mu_f}{k} \tag{5.9}$$

- $p(x,t)$: pore-fluid pressure in [MPa]
- $q(x,t)$: specific fluid discharge in [m/sec]
- k : physical permeability of the porous medium in $[m^2]$. Reservoir engineers often use the unit darcy,

$1\,\text{darcy} = 1\,\text{d} = 9.8697 \cdot 10^{-9}\,\text{cm}^2 = 9.8697 \cdot 10^{-13}\,\text{m}^2$
or
$1\,\text{md} = 9.8697 \cdot 10^{-16}\,\text{m}^2 \quad , \quad 1\text{d} = 1000\text{md}$

For a typical reservoir sandstone (Red Wildmoor):
$k = 5 \cdot 10^{-13}\,\text{m}^2 = 0.507\,\text{d} = 507.\,\text{md}$

- μ_f : viscosity of the fluid in [cP][5].

The viscosity of
1. water at 20^0C: $\mu_w = 1.002 \cdot 10^{-3}$ Pa sec
2. oil: $\mu_{oil} = 3.6 \cdot 10^{-9}$ MPa $\cdot$ sec $= 3.6 \cdot 10^{-3}$ Pa $\cdot$ sec

In CGS*:*
1. *air:* $\mu_a = 2 \cdot 10^{-4}$ gr /(cm sec) = 0.02 cP ($\rho_a = 0.00129$ gr / cm^3)
2. *water* : $\mu_w \approx 10^{-2}$ gr /(cm sec) = 1. cP ($\rho_w = 1.$ gr / cm^3)
3. *oil* : $\mu_{oil} \approx 3.6 \cdot 10^{-2}$ gr /(cm sec) = 3.6 cP ($\rho_w = 0.78$ gr / cm^3)
4. *glycerin:* $\mu_{gly} = 9$ gr /(cm sec) = 900. cP ($\rho_g = 1.265$ gr / cm^3).

Finally we remark that both density and viscosity of fluids are more or less temperature dependent.

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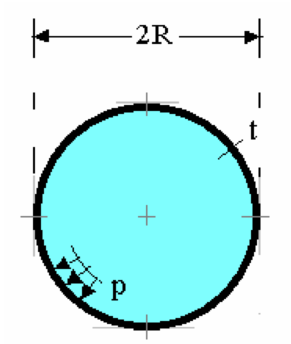
[5] $1\,\text{poise} = 1\,\dfrac{\text{gr}}{\text{cm sec}} \quad , \quad 1\,\text{cpoise} = 10^{-2}\,\text{poise}$

$1\,\text{Pa sec} = 1\,\dfrac{\text{N sec}}{\text{m}^2} = 1000\,\text{cpoise} \quad \Rightarrow \quad 1\text{cP} = 10^{-3}\text{Pa sec} = 10^{-9}\text{MPa sec}$

5.3 Flow in Elastic Tubes

We consider here an elastic tube, which at zero pressure has an inner radius R_0. Let the thickness of the elastic tube be $t << R_0$. The inner fluid-pressure is increased from initially atmospheric (set to zero as reference) to a final value p. The tube is assumed to be elastic with modulus E.



Under these assumptions a simple calculation gives that the inner radius is increased by

$$\Delta R \approx \frac{R_0^2}{Et} p$$

Accordingly the cross-sectional area of the tube changes from initially

$$A_0 = \pi R_0^2$$

to

$$A = \pi (R_0 + \Delta R)^2$$

in the pressurized state. For small changes in radius we get

$$A = A_0 + \Delta A \approx A_0 \left(1 + 2\frac{R_0 p}{Et}\right)$$

In general for an elastic tube we will have the following relationship between fluid pressure and cross-sectional area,

$$A = A_0 \left(1 + \frac{p}{K}\right) \quad , \quad K = \frac{2R_0}{Et} \qquad (5.10)$$

In the considered case we deal with a tube with variable cross-section due to its inflation-deflation, caused by pressure changes. Mass balance is expressed by the storage equation[6],

[6] cf. eq. (3.5)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{5.11}$$

Adding Darcy's law closes the problem

$$Q = -\left(\frac{A}{c}\right)\frac{\partial p}{\partial x} \quad , \ c = 8\pi\frac{\mu_f}{A^2}$$

or

$$Q = -\frac{A^2}{8\pi\mu_f}\frac{\partial p}{\partial x} \tag{5.12}$$

From equations (5.11) and (5.12) one can eliminate the discharge Q, resulting in,

$$\frac{\partial A}{\partial t} = \frac{1}{8\pi\mu_f}\frac{\partial}{\partial x}(A^2\frac{\partial p}{\partial x}) \tag{5.13}$$

From eq. (5.10) we get,

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial t} = \frac{A_0}{K}\frac{\partial p}{\partial t}$$

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial p}\frac{\partial p}{\partial x} = \frac{A_0}{K}\frac{\partial p}{\partial x}$$

Thus eq. (5.13) finally gives,

$$\frac{\partial p}{\partial t} = \frac{1}{8\pi\mu_f}\left(2A\left(\frac{\partial p}{\partial x}\right)^2 + \frac{A^2K}{A_0}\frac{\partial^2 p}{\partial x^2}\right)$$

For small changes in pressure the quadratic term in the r.h.s. of the above equation may be neglected and the coefficient in front of the second term may be linearized, resulting finally in,

$$\frac{\partial p}{\partial t} = C_p\frac{\partial^2 p}{\partial x^2} \tag{5.14}$$

where

$$C_p \approx \frac{A_0K}{8\pi\mu_f} = \frac{ER_0t}{16\mu_f}$$

Equation (5.14) is a **pressure-diffusion** equation[7]. This means that allowing for flexible pipe-walls, pressure changes are diffusing along the tube. We remark that the pressure **diffusivity coefficient** C_p has dimensions $[C_p] = L^2 T^{-1}$.

5.4 The Diffusion Equation

Given an irrigation pipe with length L and circular cross-section (E, R, t). The pressure inside the flexible pipe, obeys the pressure-diffusion equation

$$\frac{\partial p}{\partial t} = C_p \frac{\partial^2 p}{\partial x^2}$$

The initial condition for the pressure is

$$t = 0: \qquad p = p_0 \qquad \forall x \ni [0, L]$$

At time $t = 0^+$ the pressure at the entry point is increased by Δp, whereas it is kept constant at the exit point. This means that for all $t > 0$ we assume the following boundary conditions,

$$x = 0: \qquad p = p_1 = p_0 + \Delta p$$

$$x = L: \qquad p = p_0$$

We want to determine the time evolution of the pressure as function of position inside the tube and of time, $p = p(x,t)$.

First we introduce a new set of **non-dimensional** dependent and independent variables,

- non-dimensional pressure: $p^* = \dfrac{p}{p_0}$
- normalized position coordinate: $x^* = \dfrac{x}{L} \quad (0 \le x^* \le 1)$
- non-dimensional *'time factor'*: $t^* = \dfrac{t}{t_c}$, $t_c = \dfrac{L^2}{C_p}$ *(characteristic time)*

With these new variables and by avoiding for simplicity in notation the superimposed asterix, the governing equation takes the 'standard' form of a diffusion equation

[7] In the Applied Mathematics Literature such a partial differential equation is classified as a **parabolic** equation.

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} \qquad (5.15)$$

with the initial condition,

$$t = 0: \qquad p = 1 \qquad \forall x \ni [0,1]$$

and the boundary conditions,

$$x = 0: \qquad p = p_1 = 1 + \lambda \quad , \quad \lambda = \frac{\Delta p}{p_0}$$
$$x = 1: \qquad p = p_2 = 1$$

5.4.1 Fourier series solution

First we consider the steady solution,

$$\frac{\partial p}{\partial t} = 0 \quad \Rightarrow \quad \frac{d^2 p}{dx^2} = 0 \quad \Rightarrow \quad p = \overline{p} = p_1 + (p_1 - p_2)x$$

and introduce a re-normalized pressure

$$\hat{p} = \frac{p - \overline{p}}{p_1 - p_2}$$

s.t.

$$\frac{\partial \hat{p}}{\partial t} = \frac{\partial^2 \hat{p}}{\partial x^2} \quad , \quad 0 \le x \le 1$$
$$\hat{p} = h(x) = x - 1 \quad , \quad t = 0 \,, 0 \le x \le 1 \quad \text{(ic)}$$
$$\hat{p} = 0 \quad t > 0 \,, x = 0 \wedge x = 1 \quad \text{(bc)}$$

Use of the separation of variables technique

$$\hat{p}(x,t) = f(x)g(t)$$

with

$$\frac{\partial \theta}{\partial t} = f(x)\frac{dg}{dt} = f\dot{g}, \qquad \dot{g} = \frac{dg}{dt}$$
$$\frac{\partial \theta}{\partial x} = \frac{df}{dx}g(t) = f'g \,, \qquad f' = \frac{df}{dx}$$

results in

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2} \quad \Rightarrow \quad f\dot{g}=f''g \quad \Rightarrow \quad \frac{f''(x)}{f(x)}=\frac{\dot{g}(t)}{g(t)}=-A^2=\sigma\tau\alpha\theta.$$

Thus the original diffusion equation degenerates into two o.d.e.s,

$$f''+A^2f=0 \quad \Rightarrow \quad f=a\cos(Ax)+b\sin(Ax)$$
$$\dot{g}+A^2g=0 \quad \Rightarrow \quad g=c\exp\left(-A^2t\right)$$

where a,b,c are integration constants. The general solution is of the form,

$$p=\left(a\cos(A\,x)+b\sin(A\,x)\right)\exp\left(-\,A^2\,t\right)$$

From the first b.c. we get, $\forall t>0$

$$x=0: \quad p=0 \quad \Rightarrow \quad a=0$$

and from the second b.c

$$x=1: \quad \hat{p}=0 \quad \Rightarrow \quad b\sin(A)=0 \quad , \quad b\neq 0 \quad \Rightarrow \quad \underline{A=n\pi} \quad (n=1,2,\ldots)$$

Thus we get solutions of the form

$$\hat{p}_n=b_n\sin(n\pi x)\exp\left(-n^2\pi^2t\right)$$

Since the original governing equation is linear, the general solution is a Fourier series over all positive integers

$$\hat{p}=\sum_{n=1}^{\infty}b_n\sin(n\pi x)\exp\left(-n^2\pi^2t\right)$$

The Fourier series coefficients are determined from the i.c. for $t=0$:

$$h(x)=\sum_{n=1}^{\infty}b_n\sin(n\pi x) \text{ and } 0\leq x\leq 1 \qquad (*)$$

$$b_n=2\int_0^1 h(x)\sin(n\pi x)dx$$

Indeed by multiplying eq. (*) by $\sin(m\pi x)$ and by integrating we get,

$$\int_0^1 \sin(m\pi x)\,h(x)\,dx = \int_0^1 \sin(m\pi x)\sum_{n=1}^{\infty} b_n \sin(n\pi x)\,dx = \sum_{n=1}^{\infty} b_n \int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx$$

With $h(x) = x - 1$, the l.h.s. of this equation becomes

$$\int_0^1 (x-1)\sin(m\pi x)\,dx = \left(\frac{\sin(m\pi x)}{(m\pi)^2} - \frac{x\cos(m\pi x)}{m\pi}\right)_0^1 - \left(-\frac{\cos(m\pi x)}{m\pi}\right)_0^1$$
$$= \left(0 - \frac{\cos(m\pi)}{m\pi}\right) + \left(\frac{\cos(m\pi)}{m\pi} - \frac{1}{m\pi}\right) = -\frac{1}{m\pi}$$

By using the orthogonality condition

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} = 0 & m \neq n \\ = \frac{1}{2} & m = n \end{cases} = \frac{1}{2}\delta_{mn}$$

we obtain the expression for the Fourier series coefficients,

$$-\frac{1}{m\pi} = \sum_{n=1}^{\infty} b_n \frac{1}{2}\delta_{mn} \quad \Rightarrow \quad b_n = -\frac{2}{n\pi}$$

and with that

$$h(x) \approx -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}$$

This results into the following representation of the solution

$$\hat{p} \approx -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)\exp\left(-n^2\pi^2 t\right)$$

If we integrate this expression we obtain the average value of the pressure as a function of time

$$<\hat{p}> = \int_{x=0}^{x=1}\hat{p}(x,t)\,dx \approx -4\sum_{n=1}^{\infty}\frac{1}{((2n-1)\pi)^2}\exp\left(-((2n-1)\pi)^2 t\right)$$

In particular for $t = 0$ we get,

$$< \hat{p}(0) > \approx -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = -\frac{4}{\pi^2}\frac{\pi^2}{8} = -\frac{1}{2}$$

5.4.2 Finite difference integration scheme

The diffusion equation may be solved numerically according to the following algorithm:

1. The dimensionless spatial coordinate and time factor are discretised:

$$x_{n+1} = x_n + \Delta x \quad , \quad t^{m+1} = t^m + \Delta t$$

2. The interval $[0.,1]$ is divided into N equal subintervals:

$$\Delta x = \frac{1}{N}$$

3. The time increment is chosen as:

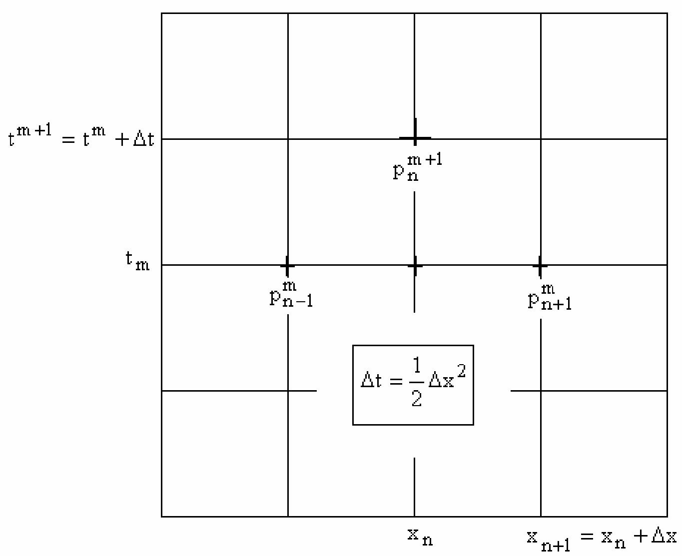
$$\Delta t = \frac{1}{2}\Delta x^2$$

4. With the notation,

$$p_n^m = p(x_n, t_m)\text{ , etc. ,}$$

the governing equation yields the following recurrent equation

$$p_n^{m+1} = p_n^m + \frac{\Delta t}{\Delta x^2}\left(p_{n+1}^m - 2p_n^m + p_{n-1}^m\right) = \frac{1}{2}\left(p_{n+1}^m + p_{n-1}^m\right)$$



5.4.3 Exercise

Evaluate numerically and compare both integration methods outlined above for the standard diffusion problem,

$$\frac{\partial \hat{p}}{\partial t} = \frac{\partial^2 \hat{p}}{\partial x^2} \quad , \quad 0 \le x \le 1$$

$$\hat{p} = h(x) = x - 1 \quad , \quad t = 0 \, , 0 \le x \le 1 \quad \text{(ic)}$$

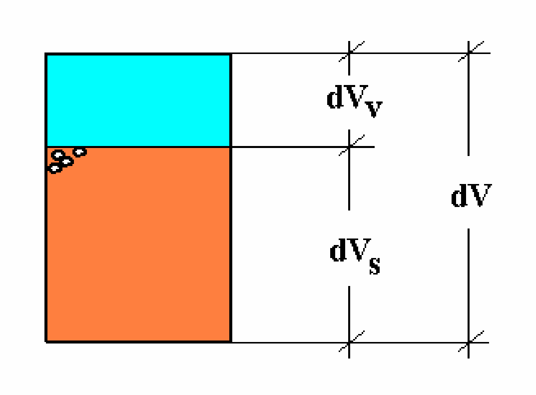
$$\hat{p} = 0 \quad t > 0 \, , x = 0 \wedge x = 1 \quad \text{(bc)}$$

5.5 Pore Pressure Diffusion

We consider a two-phase porous medium consisting of a solid skeleton (e.g. grains) and fluid, that fills completely the pore-space. We assume that both constituents are incompressible, $\rho_S = \text{const.}$, $\rho_W = \text{const.}$, and we define **partial densities** and **partial velocities** for the two phases[8]:

1. Solid phase: $\rho_1 = (1-\phi)\rho_S$, $v^{(1)}$

2. Fluid phase: $\rho_2 = \phi\rho_W$, $v^{(2)}$



According to eq. (3.6bis) mass balance for each species separately is given by the following equation,

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(\rho_\alpha v^{(\alpha)}\right) = 0 \quad , \quad (\alpha = 1{,}2)$$

For the evaluation of the above mass balance equations we define the following quantities:

- Volumetric strain rate of the solid phase: $\dot{\varepsilon} = \dfrac{\partial v^{(1)}}{\partial x}$

- The relative specific discharge : $q_r = \phi\left(v^{(2)} - v^{(1)}\right)$

---
[8] cf sect. 3.6

- The material time derivative with respect to the constituent (α) defined through the differential operator:

$$\frac{D^{(\alpha)}}{Dt} = \frac{\partial}{\partial t} + v^{(\alpha)} \frac{\partial}{\partial x}$$

With these definitions the mass balance equations for the two phases result finally to the following two well-known equations, a) an equation that expresses the fact that for incompressible constituents all volume changes of the solid skeleton are due to changes in porosity,

$$\dot{\varepsilon} = \frac{1}{1-\phi} \frac{D^{(1)}}{Dt} \phi \tag{5.15}$$

and b) that these porosity changes provide space for the 'storage' of fluid

$$\frac{1}{1-\phi} \frac{D^{(1)}}{Dt} \phi = -\frac{\partial q_r}{\partial x} \tag{5.16}$$

Eq. (5.14) is analogous to the storage equation (3.5).

In analogy to eq. (5.10) we make now the simplest possible constitutive assumption that porosity is affected by the pore-fluid pressure, e.g. by a linear law

$$\left(\frac{\partial \phi}{\partial p}\right) = \frac{1}{K_\phi} : \text{const.} \tag{5.17}$$

K_ϕ is a compressibility coefficient that accounts for the inflation of the interconnected void space due to increases in pore-fluid pressure. From eqs. (5.16) and (5.17) we get:

$$-\frac{\partial q_r}{\partial x} = \frac{1}{K_\phi (1-\phi)} \frac{\partial p}{\partial t} = c_m \frac{\partial p}{\partial t} \tag{5.18}$$

In the considered case of a deformable porous medium, Darcy's law is suitably modified and it is expressed in terms of the (objective) relative specific discharge,

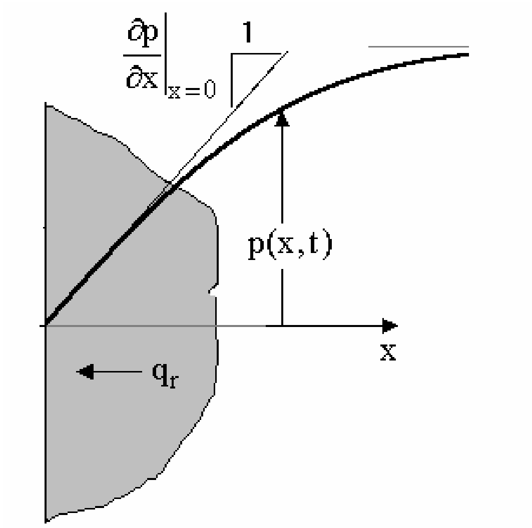
$$q_r = -\left(\frac{k}{\mu_f}\right) \frac{\partial p}{\partial x}$$

If we combine eq. (5.18) with the generalized Darcy's law we get the well-known *pore-pressure diffusion equation* of reservoir engineering,

$$\frac{\partial p}{\partial t} = c_m \frac{\partial}{\partial x}\left(\frac{k(\phi)}{\mu_f}\frac{\partial p}{\partial x}\right) \qquad (5.19)$$

5.6 Reservoir Depletion-Rate Estimate

In order to estimate the depletion of an oil or earth-gas production well we consider as a first approximation a half-space problem.



For simplicity we assume constant permeability and introduce the following dimensionless independent variables,

$$x^* = \frac{x}{L} \quad , \quad t^* = \frac{C_p t}{L^2}$$

where L is an arbitrary length scale and C_p is the pore-pressure diffusivity coefficient,

$$C_p = \frac{c_m k}{\mu_f}$$

After re-scaling above pore-pressure diffusion equation (5.19) assumes the 'standard' form

$$\frac{\partial p}{\partial t^*} = \frac{\partial^2 p}{\partial x^{*2}}$$

This equation is now solved in the half space $x \geq 0$ with the following initial and boundary conditions:

$t = 0 : p = p_0$ (i.c.)

$x = 0 : p = 0$ (b.c.)

In diffusion problems, time scales as length square; this is obvious from elementary dimensional analysis considerations. Since the half-space problem is free of any geometric length, we seek for self-similar solutions of the form,

$$p = \hat{p}(\xi) \quad , \quad \xi = \frac{x^*}{\sqrt{t^*}} = \frac{x}{\sqrt{C_p t}}$$

Inserting the above ansatz into the governing standard diffusion equation, yields the following o.d.e.,

$$2\frac{d^2\hat{p}}{d\xi^2} + \xi\frac{d\hat{p}}{d\xi} = 0$$

Thus we get,

$$\frac{d\hat{p}}{d\xi} = C_1 e^{-\frac{\xi^2}{4}} \quad ?a? \quad \hat{p} = C_1\int_0^{\xi} e^{-\frac{s^2}{4}} ds + C_2$$

This means that the solution is given in terms of the function

$$\hat{p} = \mathrm{erfc}\left(\frac{\xi}{2}\right) = 1 - \mathrm{erf}\left(\frac{\xi}{2}\right)$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2} ds = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{k!(2k+1)} \quad (|z| < \infty)$$

is the so-called error function[9],

As can be seen from the graph the error function is monotonously increasing with $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\infty) = 1$ since

$$\int_0^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$$

Since the variable ξ is independent of any scaling length, above result is proving the of existence of similarity solutions.

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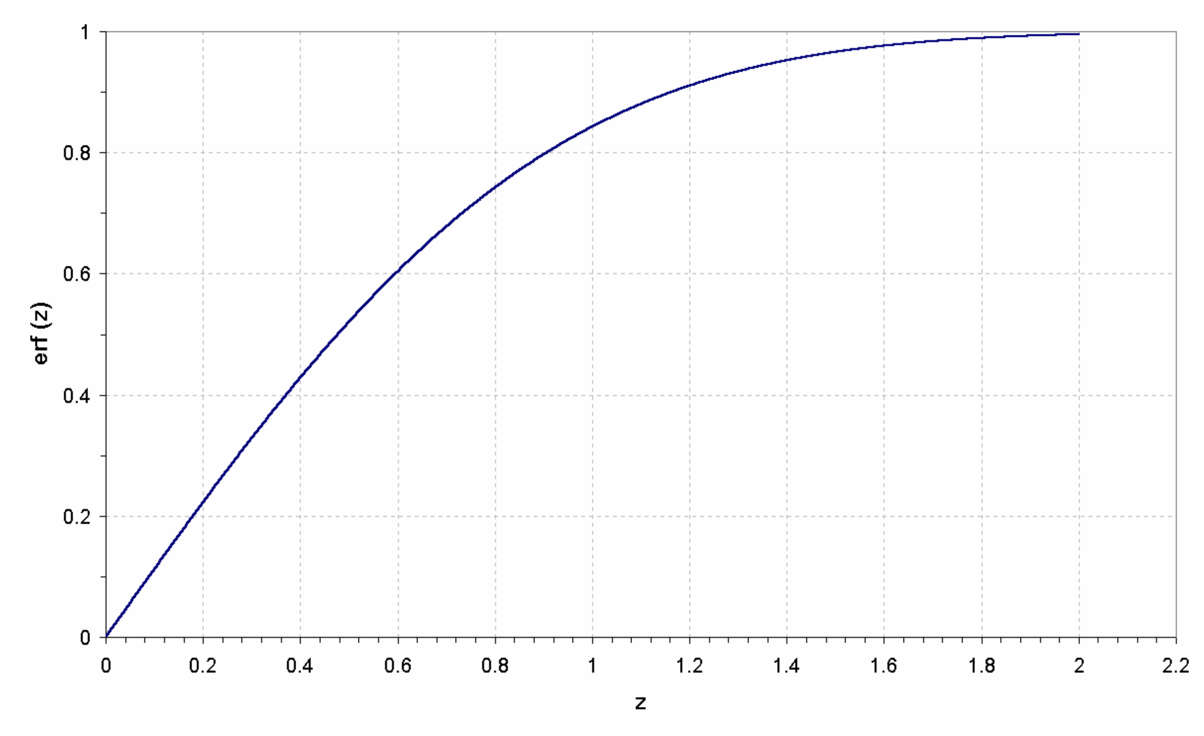
[9] ? . Abramowitz and I. Stegun. *Handbook of Mathematical Functions*, Sect. 7, Dover,1965.



From this solution follows that the pressure gradient at the exit point ($x = 0$) is

$$\left.\frac{\partial p}{\partial x}\right|_{x=0} = \frac{p_0}{\sqrt{\pi C_p t}}$$

According to Darcy's law this means that the flow rate at the half-space surface is going to fade as the square-root of time,

$$q_r(0,t) \propto \frac{1}{\sqrt{t}}$$

Above formula gives a fair impression of the depletion-rate of a production well and may be used to provide predictions for the flow rate in future times based on early-time measurements,

$$\frac{q_r(0,t_2)}{q_r(0,t_1)} = \sqrt{\frac{t_1}{t_2}}$$

6 ENERGY BALANCE

6.1 The 1st Law of Thermodynamics

The principle of energy conservation is an expression of the 1st Law of Thermodynamics. Energy is a physical quantity with dimensions of mechanical work. Mechanical energy in SI-system is measured in Joule

$$1 \text{ Joule} = 1 \text{ Nm}$$

Thermal energy is measured in cal.

Based on the 1st Law we may define a quantity j, the so-called mechanical equivalent of heat such that

$$j = 4.2 \, \frac{\text{Joule}}{\text{cal}}$$

This quantity j appears in the various expressions of the 1st Law of Thermodynamics, whenever mechanical work is converted to heat or the other way around.

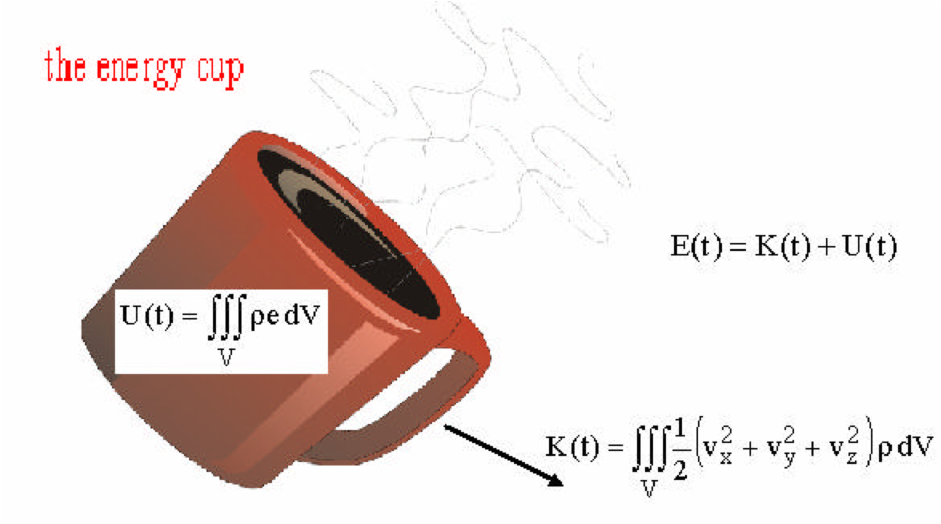
Let a continuous material body **B,** which at time t has the configuration $C^{(t)}$ that occupies the volume V with the boundary ∂V. We assume that the total **energy** $E(t)$ of the considered body material body **B** consists of two parts,

$$E(t) = K(t) + U(t) \tag{6.1}$$

where

a) $K(t) = \iiint\limits_V \frac{1}{2}\left(v_x^2 + v_y^2 + v_z^2\right)\rho\, dV$ is the total **kinetic energy,** and

b) $U(t) = \iiint\limits_V e\rho dV$ is the total **internal energy**, which does not depend on the relative motion of the considered body with respect to the observer. In this definition e is the *specific internal energy* of the body **B**.



The 1st Law of Thermodynamics requires that the change of the total energy of the material body **B** be due to two factors:

a) the power $W^{(e)}$ of all external forces acting on B in $C^{(t)}$,

b) the non-mechanical energy Q which is supplied per unit time to **B** from the exterior domain:

$$\boxed{\dot{E} = W^{(e)} + Q} \tag{6.2}$$

We may now combine eqs. (6.1) and (6.2), by eliminating $\dot{E}$:

$$\underline{\dot{U} + \dot{K} = W^{(e)} + Q} \tag{6.3}$$

We may now utilize Reynold's transport theorem and compute the material time derivatives of the body's kinetic and internal energy. If mass is preserved, then we get:

$$\dot{U} = \iiint_V \rho \dot{e} dV$$

$$\dot{K} = \iiint_V \rho\left(\dot{v}_x v_x + \dot{v}_y v_y + \dot{v}_z v_z\right) dV \tag{6.4}$$

To show the validity of these results we may consider again for simplicity a one-dimensional continuum between the sections $x = a$ and $x = b$, with

$$a = \chi^L(\alpha, t) = \hat{a}(t) \quad , \quad b = \chi^L(\beta, t) = \hat{b}(t)$$

In an Eulerian description the material points of the continuum are equipped with a mass density $\rho(x,t)$, a velocity $v(x,t)$ and a specific internal energy $e(x,t)$. Then,

$$\frac{dU}{dt} = \frac{d}{dt}\int_{\hat{a}(t)}^{\hat{b}(t)} \rho e dx = \int_a^b \frac{\partial}{\partial t}(\rho e)\, dx + v\, \rho e\big|_b - v\, \rho e\big|_a = \int_a^a \left\{\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho e v)\right\} dx$$

We observe that the integrand may be written as follows,

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho e v) = \frac{\partial \rho}{\partial t} e + \rho \frac{\partial e}{\partial t} + \frac{\partial(\rho v)}{\partial x} e + \rho v \frac{\partial e}{\partial x}$$

$$= e\left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v)\right\} + \rho\left(\frac{\partial e}{\partial t} + v\frac{\partial e}{\partial x}\right)$$

If mass is preserved we see immediately the validity of eq. (6.4a). Similarly we obtain eq. (6.4b).

6.2 The Energy Balance Law

6.2.1 The power of external forces

The power of external forces acting in the volume V and on its boundary ∂V consists of the power of body forces and of the power of surface tractions:

$$\dot{W}^{(e)} = \iiint_V (f_x v_x + f_y v_y + f_z v_z) dV + \iint_{\partial V} (t_x v_x + t_y v_y + t_z v_z) dS$$

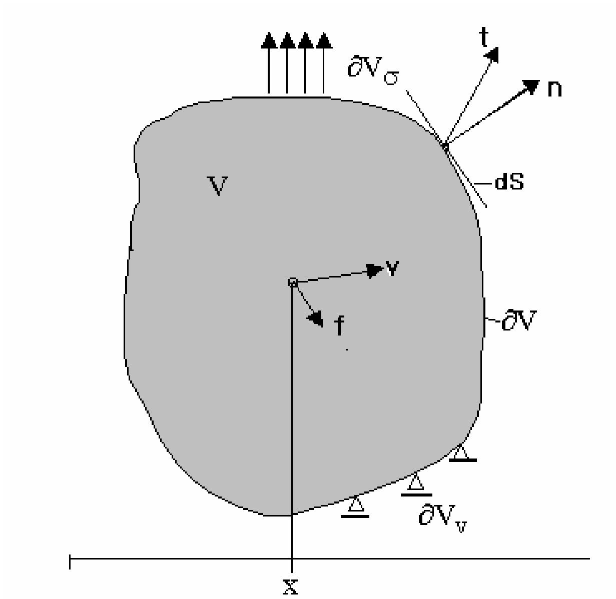
Surface tractions are given in terms of the Cauchy stress tensor **s** and of the unit outward normal vector **n** at a given point on a surface

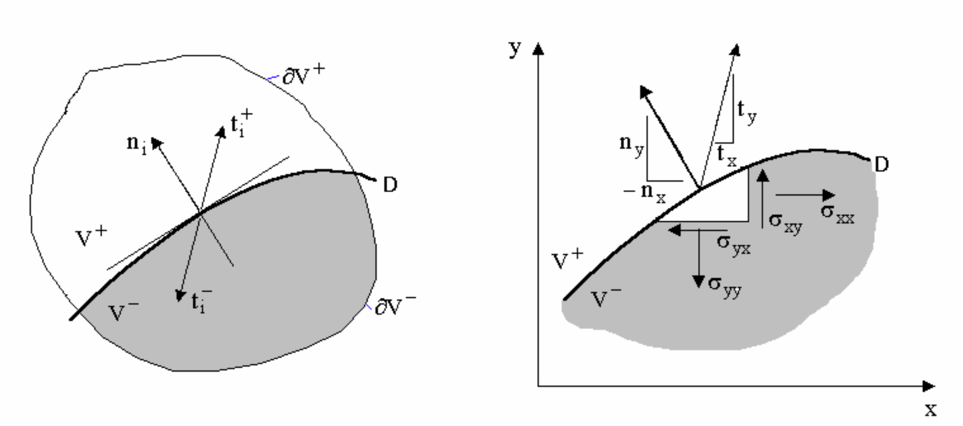
$$t_x = \sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z \quad , \ldots$$

and accordingly

$$W^{(e)} = \iiint_V (f_x v_x + f_y v_y + f_z v_z)dV + \iint_{\partial V} \left(\left(\sigma_{xx} n_x + \sigma_{yx} n_y + \sigma_{zx} n_z\right) v_x + \cdots\right) dS$$

$$= \iiint_V (f_x v_x + f_y v_y + f_z v_z)dV + \iint_{\partial V} \left(\left(\sigma_{xx} v_x + \sigma_{xy} v_y + \sigma_{xz} v_z\right) n_x + \cdots\right) dS$$



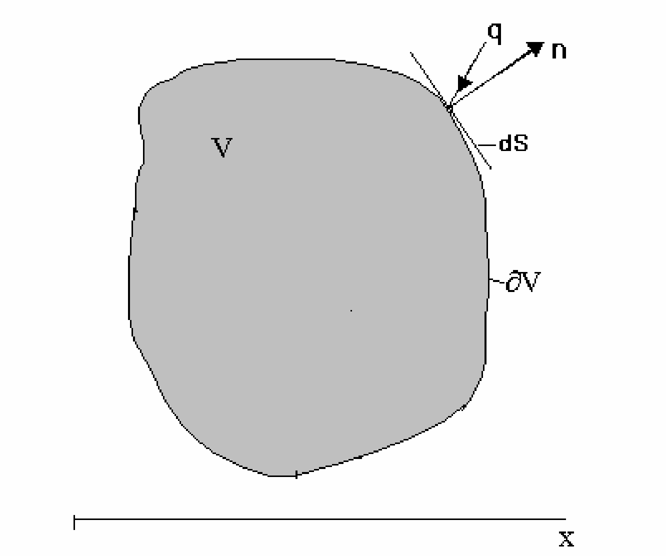


6.2.2 Heat flux

The total energy flux into V across ∂V can be expressed by *an energy flux vector* **q**, which is set positive whenever it is opposite to the unit outward normal vector n_i on the boundary ∂V of the considered volume V. Thus we may set,

$$\dot{Q} = -\iint_{\partial V} (q_x n_x + q_y n_y + q_z n_z) dS$$

For example, if non-mechanical energy transfer is only due to *heat conduction*, then **q** becomes the *heat flux vector* measured per unit surface dS in C.



Definition of the heat-flux vector

6.2.3 The Power of internal forces

In order to evaluate further the above energy balance equation (6.3) we use here the following

Theorem:
The difference between the power of external forces and the rate of kinetic energy is equal to the power of "internal forces",

$$W^{(e)} - \dot{K} = W^{(i)}$$

where

$$W^{(i)} = \iiint_V P dV$$

P is the so-called *specific stress power*

$$P = \sigma_{xx} D_{xx} + \sigma_{xy} D_{xy} + \cdots + \sigma_{zz} D_{zz}$$

and **D** is the *rate of deformation tensor*

$$D_{xx} = \frac{\partial v_x}{\partial x} \quad , \quad D_{xy} = D_{yx} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad ,\ldots$$

Proof:
This theorem follows directly from the dynamical equations

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+f_x=\rho\dot{v}_x \quad ,\dots$$

We contract them with the velocity vector $\{v_x, v_y, v_z\}$ and integrate the resulting identity over the domain V

$$\iiint_V\left(\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}+f_x-\rho\dot{v}_x\right)v_x+\cdots\right)dV=0$$

Above integral equation becomes

$$\dot{K}=\iiint_V(f_xv_x+f_yv_y+f_zv_z)dV+\iiint_V\left(\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)v_x+\cdots\right)dV=$$
$$=\iiint_V(f_xv_x+f_yv_y+f_zv_z)dV+$$
$$+\iiint_V\left(\frac{\partial}{\partial x}\left(\sigma_{xx}v_x+\sigma_{xy}v_y+\sigma_{xz}v_z\right)+\cdots\right)dV-\iiint_V\left(\sigma_{xx}\frac{\partial v_x}{\partial x}+\cdots\right)dV$$

Using Gauss' theorem the second volume integral in the above expression is transformed to the corresponding surface integral

$$\iiint_V\left(\frac{\partial}{\partial x}\left(\sigma_{xx}v_x+\sigma_{xy}v_y+\sigma_{xz}v_z\right)+\cdots\right)dV=\iint_{\partial V}\left(n_x\left(\sigma_{xx}v_x+\sigma_{xy}v_y+\sigma_{xz}v_z\right)+\cdots\right)dS$$
$$=\iint_{\partial V}(t_xv_x+t_yv_y+t_zv_z)dS$$

The third volume integral is simplified further by decomposing the velocity gradient into its symmetric and anti-symmetric part

$$\frac{\partial v_i}{\partial x_j}=D_{ij}+W_{ij} \quad (i,j=x,y,z)$$

where

- $D_{xx}=\frac{\partial v_x}{\partial x} \quad , \quad D_{xy}=D_{yx}=\frac{1}{2}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right) \quad ,\dots$
- $W_{xx}=\frac{\partial v_x}{\partial x} \quad , \quad W_{xy}=-W_{yx}=\frac{1}{2}\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right) \quad ,\dots$

Since the stress tensor is symmetric the second volume integral above becomes equal to the so-called power of internal forces,

$$\iiint_V \left(\sigma_{xx} \frac{\partial v_x}{\partial x} + \cdots \right) dV = W^{(i)}$$

Thus

$$\dot{K} = \iiint_V (f_x v_x + f_y v_y + f_z v_z) dV + \iint_{\partial V} (t_x v_x + t_y v_y + t_z v_z) dS - W^{(i)}$$

or

$$\underline{W^{(e)} - W^{(i)} = \dot{K}} \qquad (6.5)$$

q.e.d.

The energy equation (6.5) is independent of the energy balance law (6.3) since it is a result of the definitions for the power of internal and external forces acting on a body, the definition of the kinetic energy of a body and the balance law of linear momentum. In particular, in case of constant kinetic energy it follows from this theorem that the power of internal forces is equal to the power of external forces,

$$\dot{K} = 0 \quad \Leftrightarrow \quad W^{(e)} = W^{(i)}$$

6.2.4 The energy balance equation

The part of the mechanical power of the external forces that does not become kinetic energy a) is identified as the power of internal forces

$$W^{(i)} = W^{(e)} - \dot{K}$$

and b) according to the 1st Law is partially stored as internal energy and partially is dissipated into heat

$$\underline{W^{(i)} = \dot{U} - Q} \qquad (6.6)$$

Or explicitly

$$\iiint_V P dV = \iiint_V \rho \dot{e} dV - \iint_{\partial V} (q_x n_x + q_y n_y + q_z n_z) dS$$

By using Gauss' theorem this energy equation is written as

$$\iiint_V P dV = \iiint_V \rho \dot{e} dV - \iiint_V \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) dV$$

or in *local form*

$$\boxed{\rho \dot{e} = P - \operatorname{div} \vec{q}} \qquad (6.7)$$

6.3 The Heat Equation

In order to give an example of application of the energy-balance law (6.7) we assume first that the rate of deformation is decomposed additively into two parts

$$D_{xx} = D^e_{xx} + D^p_{xx} \quad ,\dots$$

By computing the stress power

$$P = \sigma_{xx} D^e_{xx} + \cdots + \sigma_{xx} D^p_{xx} + \cdots$$

we identify $\mathbf{D}^e$ as the **"elastic deformation"**, which is responsible for the mechanical energy **stored** in the material

$$\dot{w}^{(e)} = \sigma_{xx} D^e_{xx} + \sigma_{xy} D^e_{xy} + \cdots + \sigma_{zz} D^e_{zz}$$

and $\mathbf{D}^p$ as the **"plastic deformation"** that corresponds to that part of the mechanical energy, which is **dissipated** in heat,

$$D = \sigma_{xx} D^p_{xx} + \sigma_{xy} D^p_{xy} + \cdots + \sigma_{zz} D^p_{zz}$$

Secondly we assume that in a first approximation the rate of the specific internal energy depends on the changes in temperature θ and on the rate of 'elastic' deformation

$$\rho \dot{e} \approx \rho j\, c\, \dot{\theta} + \dot{w}^{(e)}$$

In this expression c is the specific heat of the material.

Accordingly the energy balance law, eq. (6.7) becomes,

$$\rho j\, c\, \dot{\theta} = -\operatorname{div} \vec{q} + D$$

The energy balance law is further specialized by adopting Fourier's law of heat conduction[1],

[1] J.B. Fourier, *Theorie analytique de la chaleur*, Paris, 1822.

$$q_x = -jk^F \frac{\partial \theta}{\partial x} \tag{6.8}$$

where k^F is Fourier's **coefficient of thermal conductivity**. For example for:

- *water* $k_w^F = 0.0012 \frac{cal}{^oC\ cm\ s}$

- *sand* $k_s^F = 0.00093 \frac{cal}{^oC\ cm\ s}$

With Fourier's gradient law, the energy balance equation (11) becomes,

$$\dot{\theta} = \kappa \nabla^2 \theta + \frac{1}{\rho j\, c} D \tag{6.9}$$

where

$$\kappa = \frac{k_F}{\rho j\, c}$$

is Kelvin's coefficient of thermal diffusivity. For example for:

- *water* $\kappa_w = 0.0012 \frac{cm^2}{s}$

- *sand* $k_s^F = 0.0026 \frac{cm^2}{s}$

Equation (13) is the law that describes the coupled phenomenon of **heat conduction** and **heat generation**.

In a one-dimensional setting and by evaluation the material time derivative of the temperature field we get,

$$\frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{\partial x} = \kappa \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\rho j\, c} D$$

We remark that if convective terms are negligible, we deal mainly with temperature diffusion-generation

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{\rho j\, c} D \tag{6.10}$$

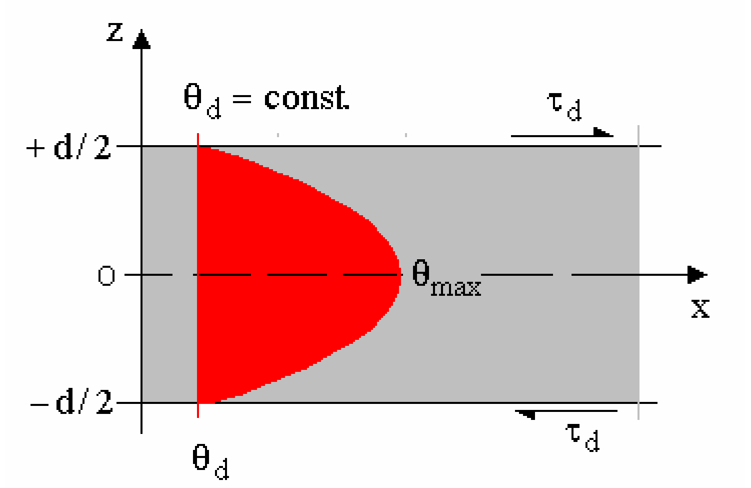
In the extreme case where diffusion is negligible, then the convective terms cannot be neglected, and the problem is one of heat-convection-generation,

$$\frac{\partial \theta}{\partial t} = -v\frac{\partial \theta}{\partial x} + \frac{1}{\rho j\, c} D$$

6.4 Steady Shear and Thermal Run-Away[2]

We consider now the steady shear deformation of a 'long' shear-band of water-saturated clayey gauge material. We assume that the considered shear-band has the thickness d, and that the various mechanical fields do not vary in its "long" x-direction; they may vary in the "short", z-direction, e.g. $\theta = \theta(z)$. Moreover the shear-band material is assumed to be at a 'critical state', deforming thus isochorically,

$$v_x = v(z) \quad , \quad v_y = v_z = 0$$



For steady, isochoric shear deformation, mass balance together with Darcy's law results to a constant pore-pressure profile across the shear-band, $p = \text{const}$. Similarly momentum balance results in constant shear stress across the shear-band

$$\sigma_{xz} = \tau_d = \text{const.}$$

Thus for the considered velocity field and for steady conditions the heat equation becomes,

$$jk_F \frac{d^2\theta}{dz^2} + D = 0 \tag{6.11}$$

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[2] Vardoulakis, I. (2002). Steady shear and thermal run-away in clayey gauges. Int. J. Solids Structures, **39**,3831-3844.

- **The Friction Law**

For soil-like materials, the steady-state shear stress (i.e. the *"strength"* of the gauge) is given formally by a friction law

$$\tau_d = \sigma_n' \mu$$

In this expression σ_n' is the effective stress in the sense of Terzaghi, acting normal to the shear-band. Since the pore-pressure is constant, then for constant total normal stress we get that σ_n' is also constant. The friction coefficient μ is the residual (large-strain or *"critical"*) friction coefficient of the gauge. We assume here that the material is frictionally thermo-visco-plastic; i.e. μ is both function of the shear strain-rate

$$\dot{\gamma} = \frac{\partial v}{\partial z}$$

and of temperature, $\mu = \hat{\mu}(\dot{\gamma}, \theta)$. We show below some typical results concerning the visco-plastity and thermo-plasticity of clays[3,4]. Accordingly we will here assume that the large-strain friction coefficient μ is decomposed multiplicatively in a strain-rate hardening power-law and a thermal softening exponential law

$$\mu = \mu_r(\dot{\gamma}) \cdot f(\theta) = \mu_{ref} \cdot \left(\frac{\dot{\gamma}}{\dot{\gamma}_{ref}} \right)^N e^{-M(\theta - \theta_1)}$$

Thus the shear stress is given by the following set of equations,

$$\tau_d = \tau_r(\dot{\gamma}) e^{-M(\theta - \theta_1)}$$

$$\tau_r = \tau_{ref} \left(\frac{\dot{\gamma}}{\dot{\gamma}_{ref}} \right)^N \qquad (6.12)$$

$$\tau_{ref} = \sigma_n' \mu_{ref} \quad , \quad m = \frac{M}{N}$$

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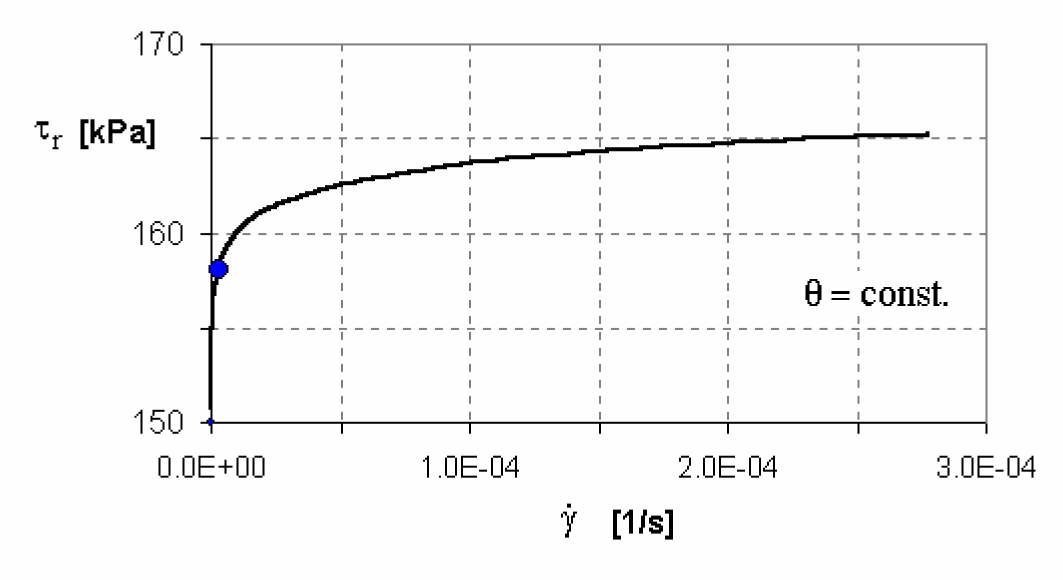
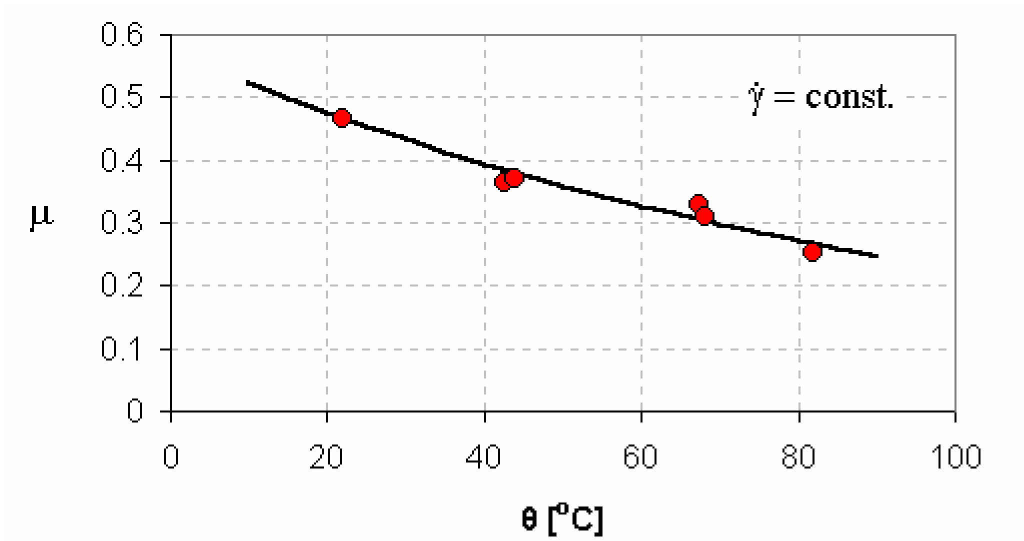
[3] Leinenkugel, H.-J. (1976), Deformations- und Festigkeitverhalten bindiger Erdstoffe. Experimentalle Ergebnisse und ihre physikalishe Deutung, Dissertation, Univ. Karlsruhe, Heft 66.

[4] Hicher, P.Y.. *Etude des proprietes mecaniques des argiles a l' aide d' essais triaxiaux, influence de la vitesse et de la temperature*, Report of the soil mechanics laboratory, Ecole Central, de Paris, 1974.



Strain-Rate sensitivity of remolded kaolin clay after Leinenkugel (1976)
($\tau_{ref} = 158.\,\mathrm{kPa}$ *,* $\dot{\gamma}_{ref} = 2.8 \cdot 10^{-6}\ \mathrm{s}^{-1}$*,* $N = 0.00101$*).*



Friction thermal softening of 'black' clay after Hicher (1974) ($\mu_r = 0.466$*,*
$\theta_1 = 22.^o\mathrm{C}$*,* $M = 0.0093\ {}^o\mathrm{C}^{-1}$*).*

We solve equation (6.12) in terms of the shear strain rate

$$\dot{\gamma} = \dot{\gamma}_{ref}\left(\frac{\tau_r}{\tau_{ref}}\right)^{\frac{1}{N}} = \dot{\gamma}_0\ e^{m(\theta-\theta_1)} \quad , \quad m = \frac{M}{N}$$

$$\dot{\gamma}_0 = \left(\frac{\mu}{\mu_{ref}}\right)^{\frac{1}{N}} \dot{\gamma}_{ref} \quad , \quad \mu = \frac{\tau_d}{\sigma_n'} = \text{const.}$$

With that we may estimate the dissipation function, by neglecting elastic strains,

$$D \approx \tau_d \dot{\gamma} = D_0 e^{m(\theta-\theta_1)} \quad , \quad D_0 = \tau_d \dot{\gamma}_0 \qquad (6.13)$$

- **Steady thermo-visco-plastic Sshearing**

With the dissipation function, equation (6.13) the heat equation (13) becomes,

$$jk \frac{d^2\theta}{dz^2} + D_0 e^{m(\theta-\theta_1)} = 0 \qquad (6.14)$$

Here we study the possibility of that the temperature at the boundaries has a constant value, equal to the ambient temperature

$$\theta(\pm d/2) = \theta_d = \text{const.}$$

The governing equation (6.14) is non-dimensionalized by introducing as new variables

$$z^* = \frac{z}{d/2} \quad , \quad \theta^* = m(\theta - \theta_1)$$

This transformation is yielding to the following non-linear, ordinary differential equation,

$$\frac{d^2\theta^*}{dz^{*2}} + \beta e^{\theta^*} = 0 \quad , \quad z^* \in [-1,1] \quad (6.15)$$

with

$$\beta = m \frac{D_0}{jk_F} \left(\frac{d}{2}\right)^2 = \frac{M}{N} \frac{\tau_d \dot{\gamma}_0}{jk_F} \left(\frac{d}{2}\right)^2 \qquad (6.16)$$

The analytical solution of the governing differential equation (6.15) can be found in mathematical textbooks[5]. This solution is given in terms of two integration constants, which are identified from the symmetry condition,

$$\theta^*(0) = \theta^*_{max} \quad , \quad \left.\frac{d\theta^*}{dz^*}\right|_{z^*=0} = 0$$

and the boundary condition

[5]Kamke, E., *Differentialgleichungen*, Vol. I, C.6., § 6.76, p. 562, Teubner, 1977.

$$\theta^*(\pm 1) = \theta_d^* = \text{const.}$$

Accordingly the solution of equation (6.15) becomes,

$$\theta^* = \theta_d^* - 2\ln\left(\frac{\cosh\left(e^{\frac{\theta_{max}^*}{2}}\sqrt{\frac{\beta}{2}}\,z^*\right)}{\cosh\left(e^{\frac{\theta_{max}^*}{2}}\sqrt{\frac{\beta}{2}}\right)}\right)$$

With this solution the boundary condition at $z^* = \pm 1$ yields

$$\theta_{max}^* = \theta_d^* + 2\ln\left(\cosh\left(e^{\frac{\theta_{max}^*}{2}}\sqrt{\frac{\beta}{2}}\right)\right) \tag{6.17}$$

Thus steady-shear is possible only if the above transcendental equation (6.17) for the maximum temperature in the middle of the shear-band θ_{max}^* has a solution. This solution depends on the value of the temperature at the shear-band boundary, θ_d^*, and on the value of the dimensionless number β, which is defined above by equation (6.16). Accordingly, before we proceed further with the discussion of the transcendental equation (6.17), we must have an estimate of the dimensionless number β.

We notice first that, according to equation (6.16), the number β combines the following information:

a) Constitutive parameters, like the ratio $m = M/N$ of the two hardening exponents and the thermal conductivity of the soil k_F.

b) The in situ stress through the 'dissipation' term:

$$D_0 = \tau_d\dot{\gamma}_0 = \tau_d\left(\frac{(\tau_d/\sigma_n')}{\mu_{ref}}\right)^{\frac{1}{N}}\dot{\gamma}_{ref}$$

c) The shear-band (fault) thickness d.

For example if we use the parameters, which are summarized above in Figures 2 and 3 and combine them with ones selected for analyzing the Vaiont landslide[6] we get:

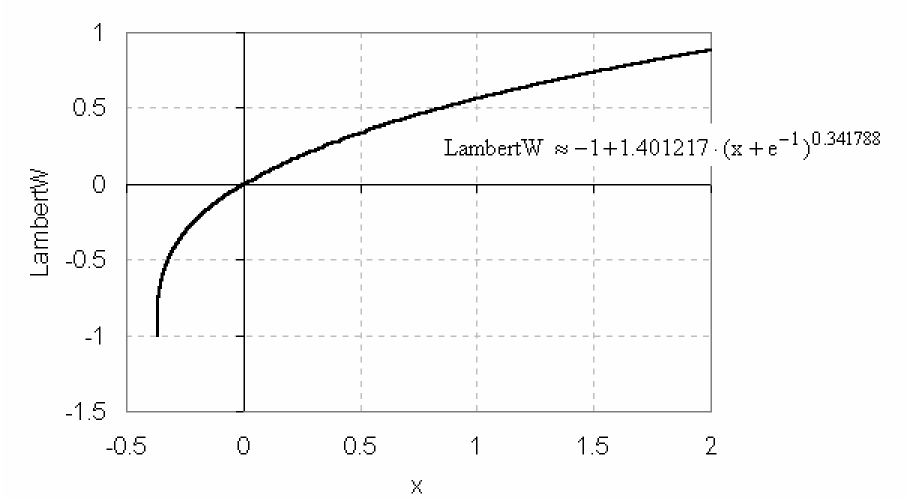
- $jk_F = 0.42 \mathrm{J}/(^{o}\mathrm{C\,m\,s})$,
- $\mu = \tan(22.3^{o})\left(\dfrac{\dot{\gamma}}{\dot{\gamma}_{ref}}\right)^{0.0101} e^{-0.0093(\theta - 22^{o}\mathrm{C})}$, $\dot{\gamma}_{ref} = 1\%/\mathrm{hr}$.
- $\sigma'_0 = 2.38\,\mathrm{MPa}$, $d = 0.1\,\mathrm{m}$.

These values are resulting to $\beta = 0.006$. Accordingly we will assume that β is a relatively small number. For small values of the dimensionless number β, the transcendental equation (6.17) becomes

$$\theta^*_d - \theta^*_{max} + \frac{\beta}{2} e^{\theta^*_{max}} + O\left(\beta^2\right) = 0 \quad (6.18)$$

The solution of this equation is given in terms of the LambertW-function[7], which satisfies the following functional relationship,

$$\mathrm{LambertW}(x)\exp\left(\mathrm{LambertW}(x)\right) = x$$



The Lambert W-function.

We notice that the LambertW-function has an order 2 branch point at $-e^{-1}$. This means that LambertW(x) is real-valued and monotonously increasing for x in the range $[-e^{-1}, +\infty]$.

Thus the only analytic at zero solution of equation (6.18) is

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[6] Vardoulakis, I. (2002). Dynamic thermo-poro-mechanical Analysis of Catastrophic Landslides. Géotechnique, **52**,No.3,157-171.

[7] c.f. R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On The Lambert W Function, Maple Share Library.

$$\theta^*_{max} = \theta^*_d - \mathrm{LambertW}\left(-\frac{\beta}{2}e^{\theta^*_d}\right) \qquad (6.19)$$

Thus according to equation (6.19) the critical value for the dimensionless ambient temperature corresponds to the branch point of the LambertW-function; i.e. for

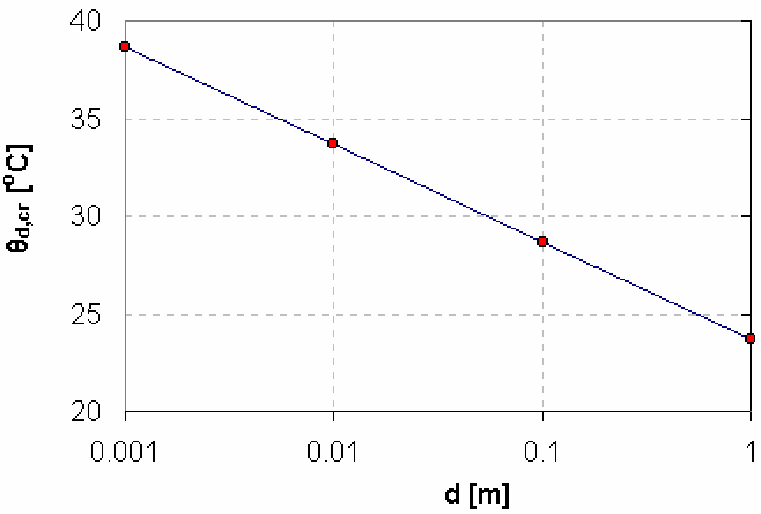
$$-\frac{\beta}{2}e^{\theta^*_{d,cr}} = -e^{-1} \quad \Rightarrow \quad \theta^*_{d,cr} = \ln\left(\frac{2}{\beta}\right) - 1 \qquad (6.20)$$

With

$$\mathrm{LambertW}\left(-e^{-1}\right) = -1 \quad \Rightarrow \theta^*_{max,cr} = \ln\left(\frac{2}{\beta}\right) \qquad (6.21)$$

This result means that above the critical temperature $\theta_{d,cr}$, steady (creeping) shear is not possible, and that the process must evolve dynamically. The corresponding phenomenon is termed in the Fluid Mechanics literature as a *thermal run-away instability* [8].

The present analysis has shown that, for thermally softening, visco-plastically hardening clays, catastrophic shear events may be caused by an increase of the ambient temperature above the problem-specific critical value $\theta_{d,cr}$, given here in dimensionless form by equation (6.20). This critical thermal-run-away ambient temperature is a logarithmically decreasing function of the shear-band thickness d.



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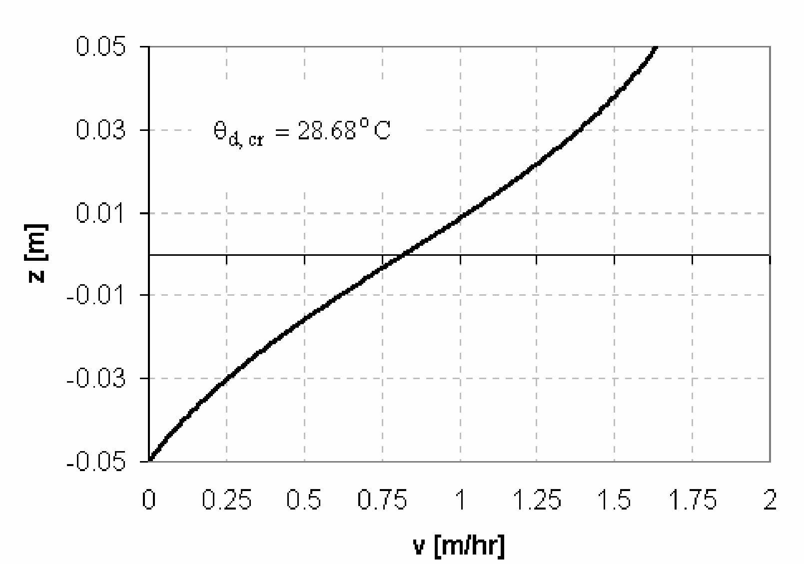
[8] Ockendon, H. and Ockendon, J.R.. *Viscous Flow*, Cambridge University Press, 1995.

The shear strain-rate in the shear-band is an exponential function of the dimensionless temperature

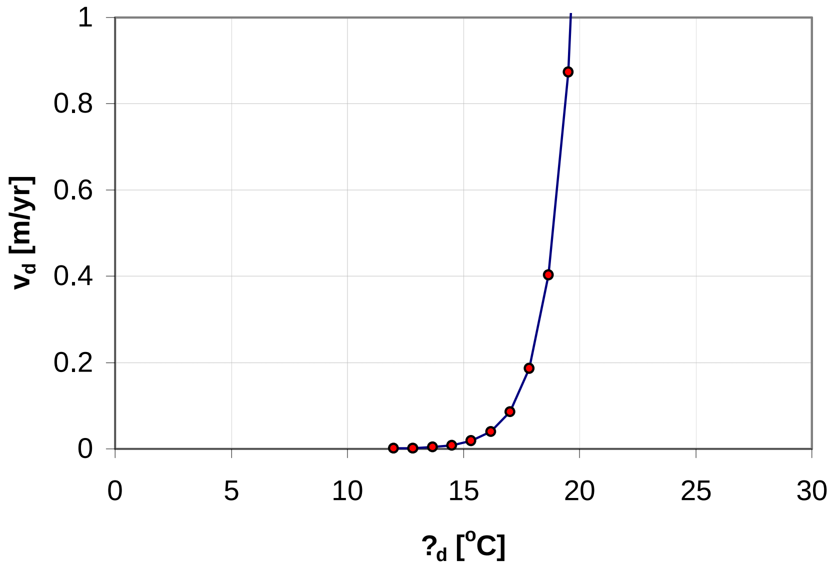
$$\dot{\gamma} = \dot{\gamma}_0 \, e^{\theta^*} \quad \Rightarrow \quad v_d = d \int_{-1}^{z^*} \dot{\gamma} d\zeta$$

The corresponding, slightly sub-linear, creep-velocity inside the shear-band profile at critical conditions and for $d = 0.1m$.



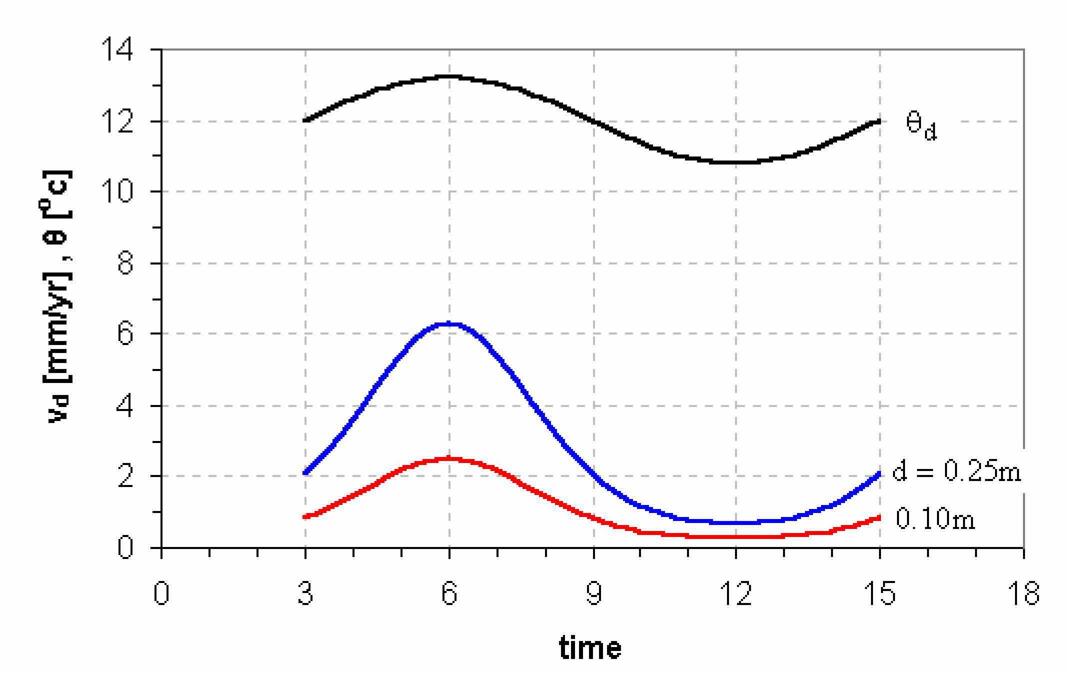
Velocity profile at critical conditions ($d = 0.1m$, $\theta_{d,cr} = 28.68^{o}C$).

We remark that the creep velocity v_d depends strongly on the ambient temperature.



Creep velocity profile at various ambient temperatures ($d = 0.1m$).

Finally, in the last figure we show the creep-velocity from resulting from a hypothetical seasonal ground temperature fluctuation for two selected values for the shear-band thickness. Such a graph may be used for estimating the thermo-mechanically effective fault thickness from creep and ground temperature measurements.



Seasonal creep (hypothetical scenario)

- **Conclusion**

The present analysis has shown that, for thermally softening, visco-plastically hardening clays, catastrophic shear events may be caused by an increase of the ambient temperature above the problem-specific critical value $\theta_{d,cr}$, given here in dimensionless form by equation (6.20).

6.5 Exercise

In analogy to the above-discussed steady simple-shear, formulate the fully dynamic problem for water, by considering that:

a) Mass balance and by assuming that the density varies with temperature but only minutely with pressure,

$$\rho_w = \bar{\rho}_w e^{3\alpha_w(\theta-\bar{\theta})} \approx \bar{\rho}_w [1 - 3\alpha_w(\theta - \theta_{ref})]$$

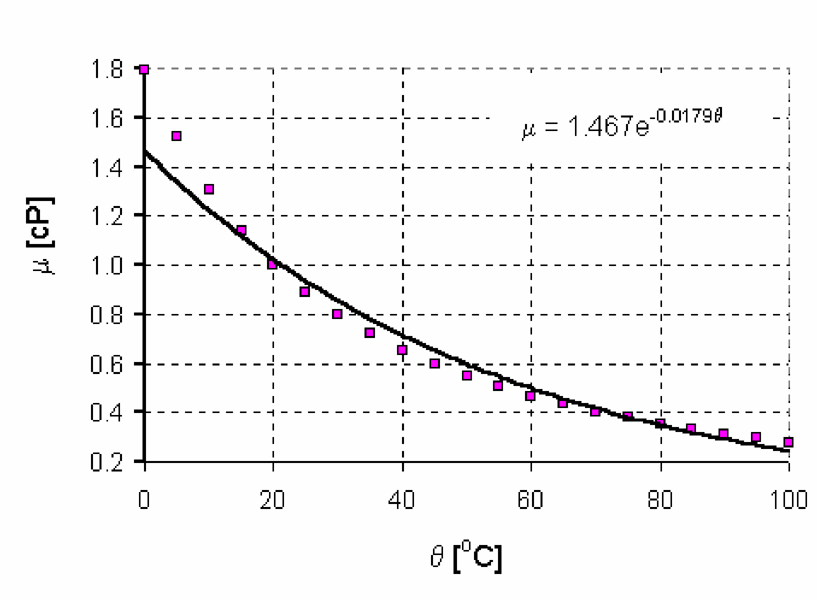
where $\alpha_w = 180 \cdot 10^{-6} \quad 1/{}^oC$, denotes the linear coefficient of thermal expansion of water.

b) <u>Balance of linear momentum</u>, by assuming Newton's law for the shear stress,

$$\tau = \mu_W \frac{\partial v}{\partial z}$$

where μ_W is the viscosity of water

$$\mu_W \approx \mu_{W,ref} e \lambda \theta$$



c) Energy balance with, $c_W = 1 \dfrac{cal}{gr\,^oC}$, $k_{F,W} = 0.0012 \dfrac{cal}{^oC\,cm\,s}$.