LARAM School 2008 (7-20 September, Ravello, Italy)

SESSION 1: Landslide analysis using approaches based on: Geology, Geotechnics and Geomechanics

Basic Geodynamics of Landslides

I. The Dynamic Slip Circle Method

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1. Problem Statement

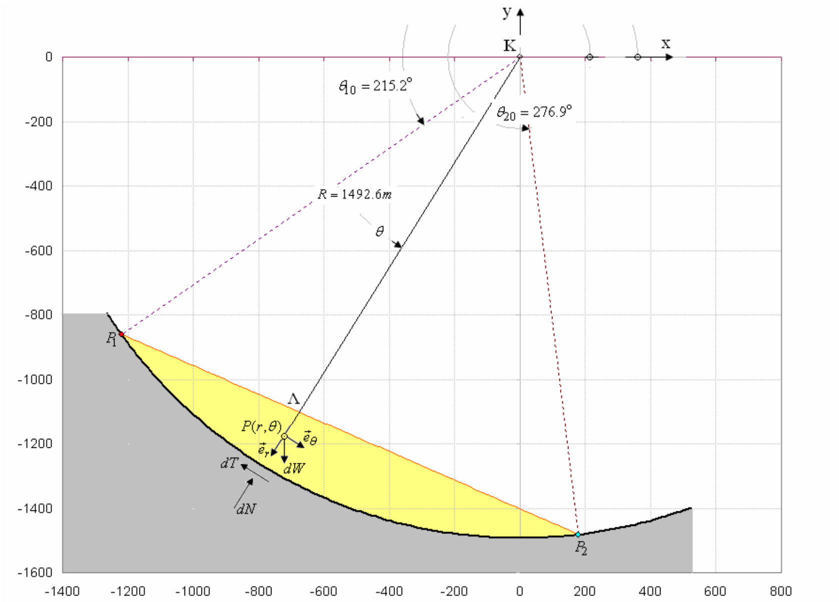

Figure 1-1: Circular arc of mobilized material, failing as a rigid body

The dynamic '*slip-circle*' method of analysis is an extension of the corresponding static method. The earth masses are assumed to rotate and accelerate like a coherent rigid body with respect to a fixed-in-space center K (Figure 1-1). This assumption is acceptable only for the initial phase of a landslide motion. The geometric characteristics of the considered circular arc are determined following a standard static analysis for the selection of the critical circular arc at incipient failure. Here we assume for simplicity an arc that cuts through a sloping ground with constant inclination and homogeneous material. The section, shown in Figure 1-1, is a simplified version of the critical Section 5 of the Vaiont slide [1], [4]. For simplicity reasons the effect of groundwater is disregarded here.

The radius of the considered circular arc is denoted by R. Initially the end-points of this arc are positioned at $P_1(R,\theta_1)$ and $P_2(R,\theta_2)$, where a polar coordinate system is used with center K and the polar angle θ measured counterclockwise, with respect to the $+x$-axis. The rotating body is rigid, thus the epicenter angle of the considered arc is constant,

$$2\alpha = \theta_2 - \theta_1 = const. \tag{1.1}$$

The polar angles of the end-points of the considered arc change with the rotation angle of the circular arc,

$$\theta_i = \theta_{i0} + \int_0^t \omega(t)dt \quad (i = 1, 2) \tag{1.2}$$

In eq. (1.2) with ω we denoted the angular velocity of rotating arc as function of time

$$\omega = \frac{d\phi}{dt} \quad \Leftrightarrow \quad \phi = \int \omega dt \tag{1.3}$$

where $\phi(t)$ is the rotation angle of the arc as function of time.

At any point $P(r,\theta)$ we introduce the local polar ortho-normal basis vectors $\vec{e}_r$ and $\vec{e}_\theta$, in radial and tangential direction, respectively. Thus at point $P(r,\theta)$ in the interior of the considered rotating circular arc with the position vector

$$\vec{R}_P = r\vec{e}_r \tag{1.4}$$

the velocity and the acceleration are,

$$\vec{v}_P = v_\theta \vec{e}_\theta \tag{1.5}$$

and

$$\vec{a}_P = a_r \vec{e}_r + a_\theta \vec{e}_\theta \tag{1.6}$$

where

$$v_\theta = r\omega \tag{1.7}$$

and

$$a_r = -r\omega^2 \tag{1.8}$$

$$a_\theta = r\frac{d\omega}{dt} \tag{1.9}$$

Let the mass of a material point be dm. As already mentioned, we assume here for simplicity that the moving earth masses are dry and that the only external force acting on them is gravity. Thus the weight of the material point is

$$d\vec{W} = dm\vec{g} = (\rho dV)\vec{g} \tag{1.10}$$

where

$$\vec{g} = -g \sin\theta \vec{e}_r - g \cos\theta \vec{e}_\theta \tag{1.11}$$

and g is the gravity acceleration,

Along the periphery of the considered failure arc reaction forces are acting (Figure 1-1). These are:

a) Normal reaction forces,

$$d\vec{N} = (-\sigma_n dS)\vec{e}_r \tag{1.12}$$

b) Tangential reaction forces,

$$d\vec{T} = (-\tau_n dS)\vec{e}_\theta \tag{1.13}$$

For a purely frictional material the shear- and normal stresses along the failure arc are related through Coulomb's friction law,

$$\tau_n = \mu\sigma_n \quad , \quad \mu = \tan\varphi \tag{1.14}$$

where φ is the Coulomb friction angle at the slip surface.

2. Estimation of the dynamic normal reaction

The normal reaction stress σ_n is restricted by the dynamic equation for the considered moving body in radial direction (i.e. in normal direction to the slip surface),

$$\int_{(V)} d\vec{W}_r + \int_{(S)} d\vec{N}_r = \int_{(m)} \vec{a}_r dm \tag{2.1}$$

Using eqs. (1.10), (1.12) and (1.8) we get,

$$\int_{(V)} \rho g(-\sin\theta)dV + \int_{(S)} (-\sigma_n)dS = \int_{(m)} \rho(-r\omega^2)dV \tag{2.2}$$

Let ℓ be the unit length in a direction perpendicular to the considered plane, then

$$dV = \ell r dr d\theta \tag{2.3}$$

and

$$dS = \ell R d\theta \tag{2.4}$$

Eq. (2.2) yields

$$R\int_{\theta_1}^{\theta_2} \sigma_n d\theta = \rho g \int_{\theta_1}^{\theta_2}\int_{R_1}^{R} (-\sin\theta) r dr d\theta + \rho\omega^2 \int_{\theta_1}^{\theta_2}\int_{R_1}^{R} r^2 dr d\theta \tag{2.5}$$

Where, as shown in Figure 1-1,

$$R_1 = (\mathrm{K}\Lambda) = R\frac{\cos\alpha}{\cos(\theta-\theta_1-\alpha)} \tag{2.6}$$

Formally eq. (2.5) is written as follows,

$$\frac{R}{\rho g}\int_{0}^{2\alpha} \sigma_n d\alpha = \int_{\theta_1}^{\theta_2}\int_{R_1}^{R} (-\sin\theta) r dr d\theta + \frac{\omega^2}{g} I_p \tag{2.7}$$

where I_p is the *polar surface moment* of inertia of the considered arc,

$$I_p = \int_{\theta_1}^{\theta_2}\int_{R_1}^{R} r^2 dr d\theta \tag{2.8}$$

We remark that the distribution of the normal reaction stress along the failure arc is undetermined. This situation is already known from the static analysis (cf. Taylor's, [2] "friction circle method"). However as explained in Vardoulakis (2002), if we want to reduce the considered moving earth body into an one-degree-of freedom "frictional pendulum", then we must assume that the normal reaction stress is distributed uniformly along the failing arc. Thus we introduce the mean value for the normal reaction stress,

$$\bar{\sigma}_n = \frac{1}{2\alpha}\int_{0}^{2\alpha} \sigma_n d\alpha \tag{2.9}$$

and with that eq. (2.7) becomes,

$$\frac{R}{\rho g}2\alpha\bar{\sigma}_n = \int_{\theta_1}^{\theta_2}\int_{R_1}^{R}(-\sin\theta)rdrd\theta + \frac{\omega^2}{g}I_p \tag{2.10}$$

Let,

$$h = R - R_1 = R\left(1 - \frac{\cos\alpha}{\cos(\theta-\theta_1-\alpha)}\right) \tag{2.11}$$

and

$$f = h_{\max} = R(1-\cos\alpha) \tag{2.12}$$

We introduce the following reference stress

$$\sigma_{ref} = \rho g f \tag{2.13}$$

the non-dimensional radial coordinate and the mean, normal reaction stress,

$$r^* = \frac{r}{R} \tag{2.14}$$

$$\bar{\sigma}_n^* = \frac{\bar{\sigma}_n}{\sigma_{ref}} \tag{2.15}$$

With eq. (2.14), eq. (2.8) becomes

$$I_p = R^3 I_p^* \tag{2.16}$$

where

$$I_p^* = \int_{\theta_1}^{\theta_2}\int_{R_1^*}^{1} r^{*2}dr^*d\theta \tag{2.17}$$

where

$$R_1^* = \frac{R_1}{R} = \frac{\cos\alpha}{\cos(\theta-\theta_1-\alpha)} \tag{2.18}$$

For the considered circular arc we get the following closed form expression for the dimensionless polar surface moment of inertia of the considered arc (Figure 2-1),

$$I_p^* = \frac{1}{3}\left(2\alpha - \frac{1}{2}\sin 2\alpha - \ln\left(\left| \tan\left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| \right) \cos^3 \alpha \right) \tag{2.19}$$

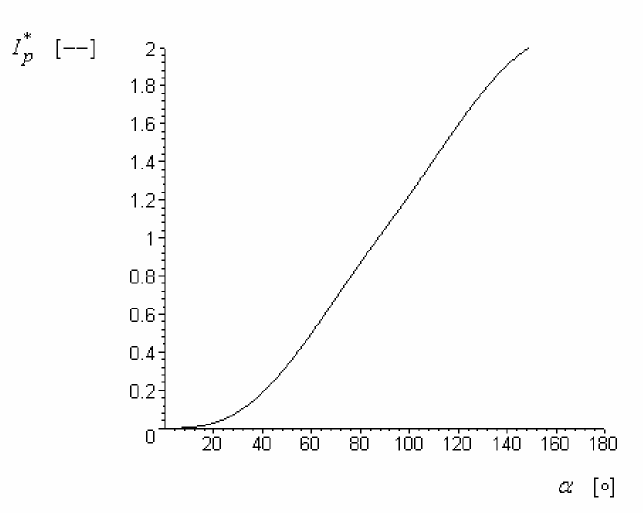

Figure 2-1: Polar moment of inertia of circular as a function of the half epicenter angle

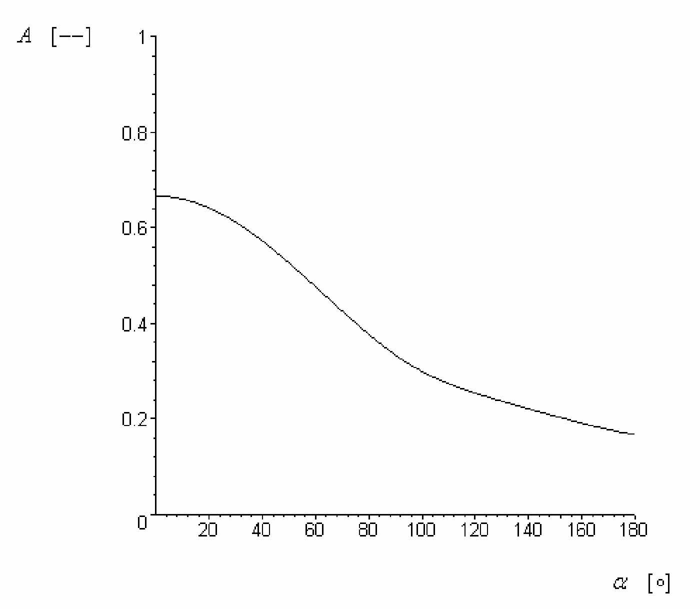

Figure 2-2: Dynamic coefficient $A(\alpha)$; eq. (2.24)

With the above notation eq (2.10) yields

$$\bar{\sigma}_n^* = \bar{\sigma}_{n,st}^* + A\omega^{*2} \tag{2.20}$$

where

$$\omega^* = \frac{\omega}{\sqrt{g/R}} \tag{2.21}$$

and

$$\bar{\sigma}^*_{n,st} = \frac{1}{2\alpha(1-\cos\alpha)} J_r(\theta_1,\theta_2) \tag{2.22}$$

$$J_r = \int_{\theta_1}^{\theta_2} \int_{r_1^*}^{1} r^*(-\sin\theta) dr^* d\theta \tag{2.23}$$

$$A = \frac{I_p^*}{2\alpha(1-\cos\alpha)} \tag{2.24}$$

The coefficient A as a function of the epicenter angle is of the order of unity, as can be seen from Figure 2-2. Thus with $A(\alpha) = O(1)$ we remark that for small values of the angular velocity ω the 2nd term in eq. (2.20), that is due to centripetal accelerations, may be neglected as being of 2nd order in ω. In this case eq. (2.20) reduces to a static equilibrium equation for the mean (normalized) normal reaction traction,

$$\bar{\sigma}^*_n = \bar{\sigma}^*_{n,st} \tag{2.25}$$

This observation justifies also the usual assumption of static equilibrium in normal direction that is made when the sliding surface is planar.

The analytic expression for the integral $J_r(\theta_1,\theta_2)$ is long and tedious and is listed in Appendix I (Sect. 5). Moreover we remark that, according to eq. (1.2), the static moment J_r, eq. (2.23), is a function of the rotation angle ϕ. For the numerical evaluation of this integral we used the values listed in Table 2-1. and the result is shown in Figure 2-3. As can be seen from this figure for small ϕ, this integral is practically constant. Accordingly we may write,

$$\bar{\sigma}^*_n = \bar{\sigma}^*_{n,0} + A\omega^{*2} \tag{2.26}$$

where

$$\bar{\sigma}_{n,0}^{*} = \frac{1}{2\alpha(1-\cos\alpha)} J_r(\theta_{10},\theta_{20}) \tag{2.27}$$

Table 2-1: Geometric and mechanical properties of the assumed circular arc (cf. Figure 1-1)

R	γ	θ_{10}	θ_{20}	2α	I_p	σ_{ref}	$\bar{\sigma}_{n,0}$	A
$[m]$	$[kN/m^3]$	$[\circ]$	$[\circ]$	$[\circ]$	$[km^3]$	$[MPa]$	$[MPa]$	$[—]$
1492.6	20	215.2	276.9	61.7	0.308	4.223	2.413	0.468

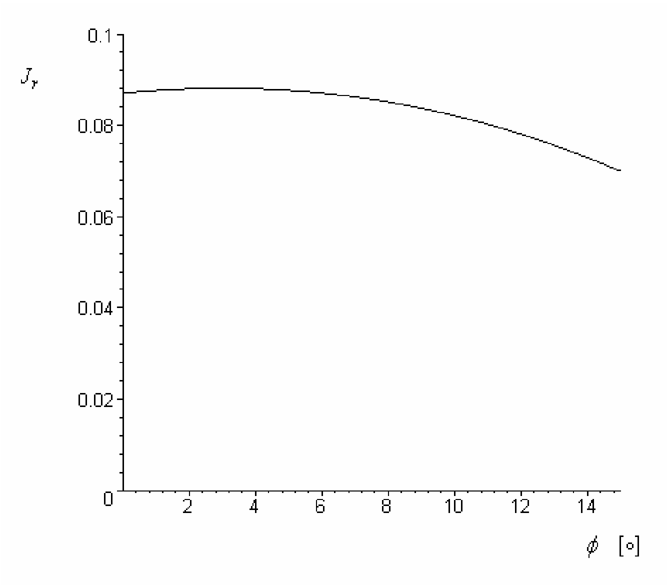

Figure 2-3: Evaluation of the integral $J_r(\theta_1,\theta_2)$, eq. (2.23), as a function of the slide rotation angle ϕ and for the values of the angular parameters of the slide as listed in Table 2-1.

3. The governing dynamic equation

The mass moment of inertia is defined as

$$\Theta = \int_{(m)} r^2 dm = \int_{(V)} r^2 \rho dV = \ell\rho \int_{\theta_1}^{\theta_2} \int_{R_1}^{R} r^3 dr d\theta \tag{3.1}$$

Balance of angular momentum requires that

$$\Theta \frac{d\omega}{dt} = M^{K} \tag{3.2}$$

where M^K is the moment with respect to the pole K of the forces acting on the considered body. These are the self weight of the body and the frictional resistance forces acting along the slip circle,

$$M^{K} = M_{W}^{K} + M_{T}^{K} = \int_{(V)} r dW_{\theta} - \int_{(S)} R dT \tag{3.3}$$

or

$$M^{K} = \int_{(V)} r\rho g \left(-\cos\theta\right) dV - R \int_{(S)} \mu \sigma_{n} dS \tag{3.4}$$

As already mentioned we assume that the normal contact stress is distributed uniformly along the failure surface. This assumption is the only compatible with an 1.d.o.f. model and for a constant friction coefficient along the failure surface [4]. Thus we set,

$$\int_{(S)} \mu \sigma_{n} dS = \mu(t) \bar{\sigma}_{n} \ell R 2\alpha \tag{3.5}$$

Notice that according to eqs. (2.20) and (2.15) we have that

$$\bar{\sigma}_{n} = \sigma_{ref} \left(\bar{\sigma}_{n,st}^{*} + A\omega^{*2} \right) \tag{3.6}$$

The first integral in eq. (3.4) becomes

$$M_{W}^{K} = \ell \rho g \int_{\theta_1}^{\theta_2} \int_{R_1}^{R} r^{2} \left(-\cos\theta\right) dr d\theta \tag{3.7}$$

Thus from eqs. (3.1) to (3.7) we get,

$$\int_{\theta_1}^{\theta_2} \int_{R_1}^{R} r^{3} dr d\theta \frac{d\omega}{dt} = g \int_{\theta_1}^{\theta_2} \int_{R_1}^{R} r^{2} \left(-\cos\theta\right) dr d\theta - \frac{R^{2} g}{\gamma} 2\alpha \mu(t) \bar{\sigma}_{n} \tag{3.8}$$

We introduce a dimensionless time factor

$$t^* = \frac{t}{t_c} \quad , \quad t_c = \sqrt{\frac{R}{g}} \tag{3.9}$$

From eq. (2.21) we get

$$\frac{d\omega}{dt} = \frac{g}{R}\frac{d\omega^*}{dt^*} \tag{3.10}$$

and with that eq. (3.8) becomes

$$\Theta^* \frac{d\omega^*}{dt^*} = J_\theta - \lambda\left(\bar{\sigma}^*_{n,0} + A\omega^{*2}\right)\mu(t^*) \tag{3.11}$$

where

$$\Theta^* = \int_{\theta_1}^{\theta_2}\int_{R_1^*}^{1} r^{*3} dr^* d\theta \tag{3.12}$$

$$J_\theta = \int_{\theta_1}^{\theta_2}\int_{R_1^*}^{1} r^{*2}\left(-\cos\theta\right) dr^* d\theta \tag{3.13}$$

$$\lambda = 2\alpha\frac{\sigma_{ref}}{\gamma R} = 2\alpha\frac{f}{R} = 2\alpha\left(1-\cos\alpha\right) \tag{3.14}$$

For the considered circular arc we get the follwing closed form expression for the dimensionless moment of inertia (Figure 3-1),

$$\Theta^* = \frac{1}{2}\left(\alpha - \frac{1}{6}\sin 2\alpha\left(1+2\cos^2\alpha\right)\right) \tag{3.15}$$

The analytic expression for the integral $J_\theta(\theta_1,\theta_2)$ is again very long and is listed in Appendix II (Sect. 6). As can be seen from Figure 3-2 for small values of the rotation angle ϕ this integral is also practically constant.

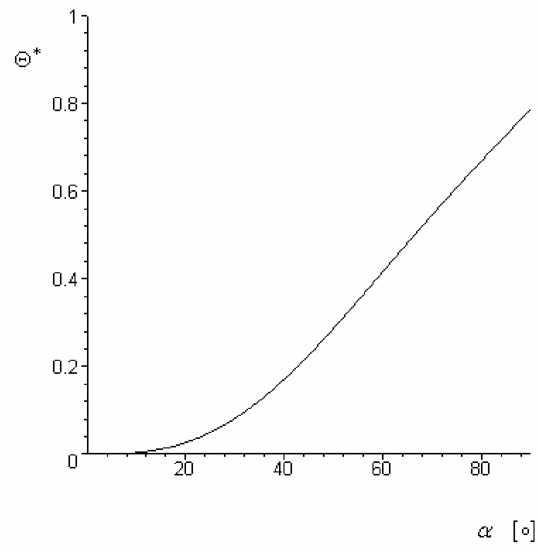

Figure 3-1: Volume polar moment of inertia as a function of the angle α

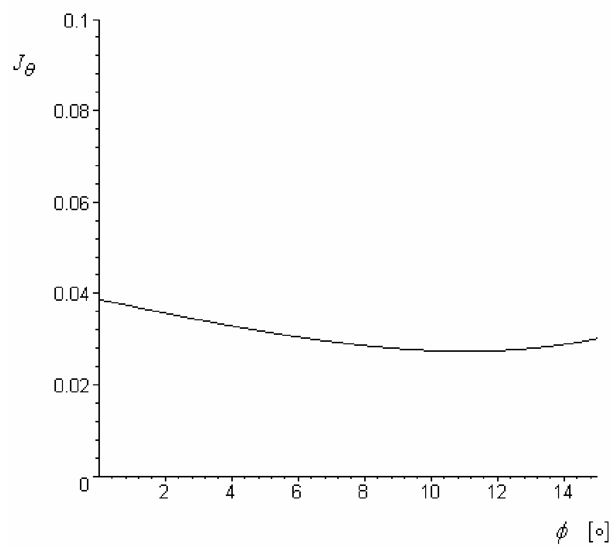

Figure 3-2: Evaluation of the integral $J_\theta(\theta_1,\theta_2)$ *, eq. (3.13), as a function of the slide rotation angle* ϕ *and for the values of the angular parameters of the slide as listed in Table 2-1*

The resulting dynamic equations is,

$$\frac{d\omega^*}{dt^*} = a - b\omega^{*2} \tag{3.16}$$

where

$$a = \frac{1}{\Theta^*}\left(J_\theta - J_{r0}\mu(t)\right) \tag{3.17}$$

$$b = \frac{\lambda A}{\Theta^*}\mu(t) \tag{3.18}$$

With the initial condition,

$$\omega^*(0) = 0 \tag{3.19}$$

The solution of the governing equation is,

$$\omega^* = \sqrt{\frac{a}{b}}\tanh\left(\sqrt{ab}t^*\right) \tag{3.20}$$

We remark that the motion starts as soon as a trigger mechanism has reduced the friction coefficient to a value that is less then the corresponding limit equilibrium value. This limit equilibrium value for the friction coefficient is given by the following equation:

$$\omega^* = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad \mu_{eq} = \frac{J_{\theta 0}}{J_{r0}} \tag{3.21}$$

We remark also that for the considered homogeneous slope the solution, eqs. (3.20) and (3.21), is independent of the density of the geo-material involved in the sliding motion.

4. Computational Example

The evaluation of eq. (3.11) is done here on the basis of the parameters listed in Table 4-1. The limit equilibrium value for the Coulomb friction coefficient is,

$$\mu_{\mathrm{eq}} = \frac{0.0387}{0.0870} = 0.44 \tag{4.1}$$

The slide motion is studied for the case of a severe reduction of the Coulomb friction angle due to shear-displacement induced friction softening. The assumed reduction of the friction angle from its equilibrium value $\varphi_{eq} \approx 24^\circ$ to the residual value

$\varphi_{res} = 10^\circ$ is taken form the experimental study of Tika & Hutrchnison [3], concerning the Vaiont slide (Figure 4-1).

Table 4-1: Dimensionless parameters entering the governing eq. (3.11); cf. Table 2-1.

Θ^*	$\sim J_r$	$\sim J_\theta$	λ	$\bar{\sigma}^*_{n,0}$	A	$\varphi_{C,eq}$
0.4376	0.0870	0.0387	0.1523	0.5713	0.468	24°

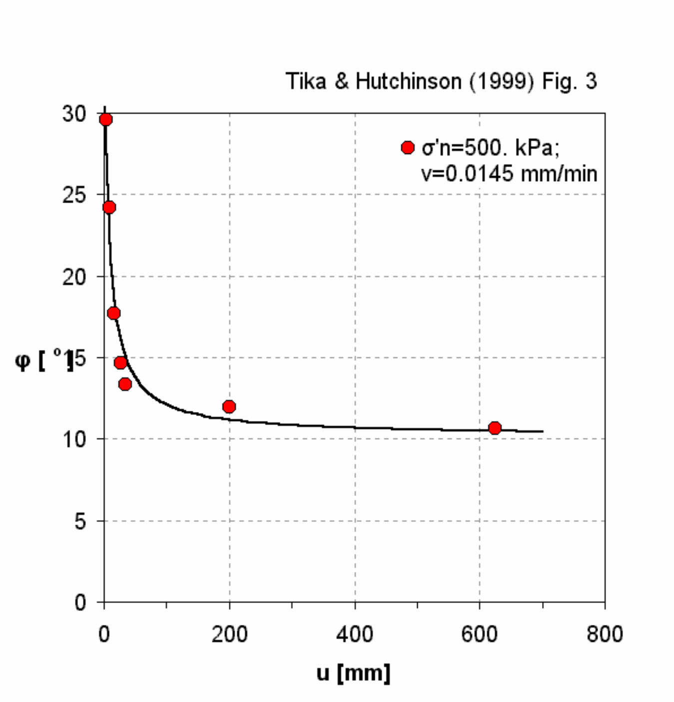

Figure 4-1: Friction-displacement softening in ring-shear tests after Tika and Hutchinson (1999).

In Figure 4-2 we plotted the evolution of the dimensionless angular velocity as a function of the dimensionless time factor. In terms dimensioned variables eq. (3.20) becomes here,

$$\omega = c_0 \tanh(c_2 t) \tag{4.2}$$

where $c_0 = 0.1104\text{sec}^{-1}$ and $c_2 = 0.00317\text{sec}^{-1}$. Figure 4-3 shows the evolution of the rotation angle of the slide

$$\phi = \frac{c_0}{c_2} \ln\left(\cosh\left(c_2 t\right)\right) \tag{4.3}$$

In Figure 4-4 we show the evolution of the slide velocity eq. (1.7),

$$v_\theta = R\omega(t) = c_1 \tanh\left(c_2 t\right) = c_1\left(c_2 t + O\left(\left(c_2 t\right)^2\right)\right) \tag{4.4}$$

where $c_1 \approx 165\text{m/sec}$. The slide displacement is then,

$$s = \int v_\theta dt = c_1 \ln\cosh\left(c_2 t\right) = c_1\left(\frac{1}{2}\left(c_2 t\right)^2 + O\left(\left(c_2 t\right)^4\right)\right) \tag{4.5}$$

We remark that these plots are restricted in the first 30 sec of the post-failure life of the slide. Although the analytic formulae derived here could be evaluated for later times, we caution to the fact that the rigid-body hypothesis eventually breaks down, rendering the present theory unrealistic.

Notably that the assumed severe reduction of the friction angle from its equilibrium value to its residual value is leading in the considered short time interval to large velocities and displacements.

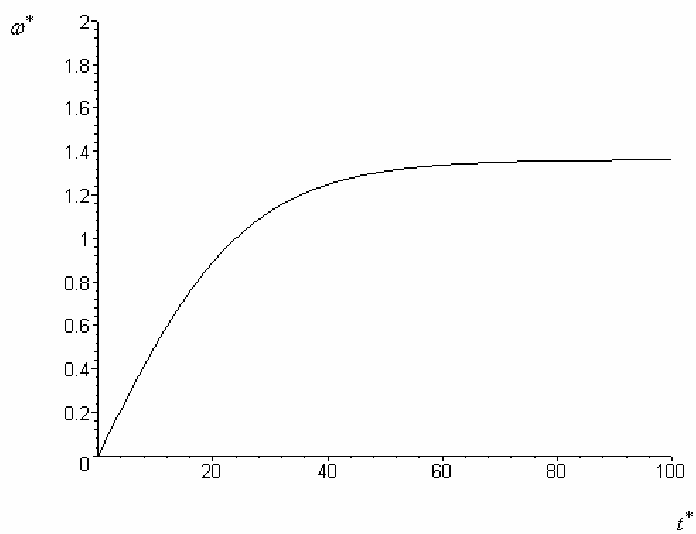

Figure 4-2: Evolution of the dimensionless angular velocity, eq. (3.20) & Table 4-1.

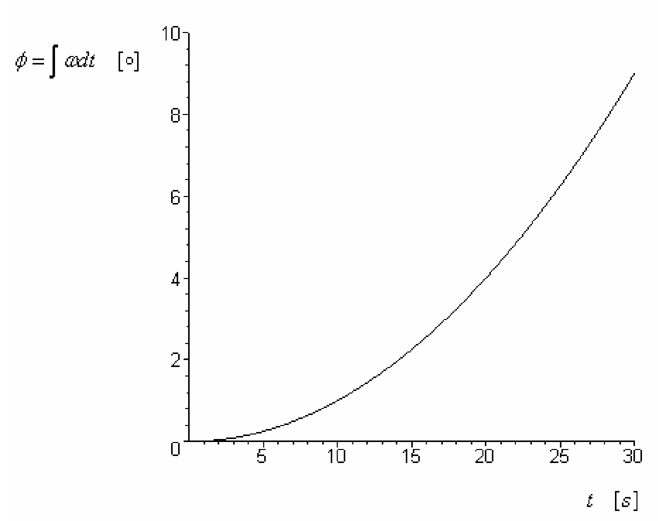

Figure 4-3: Rotation angle of the slide as function of time

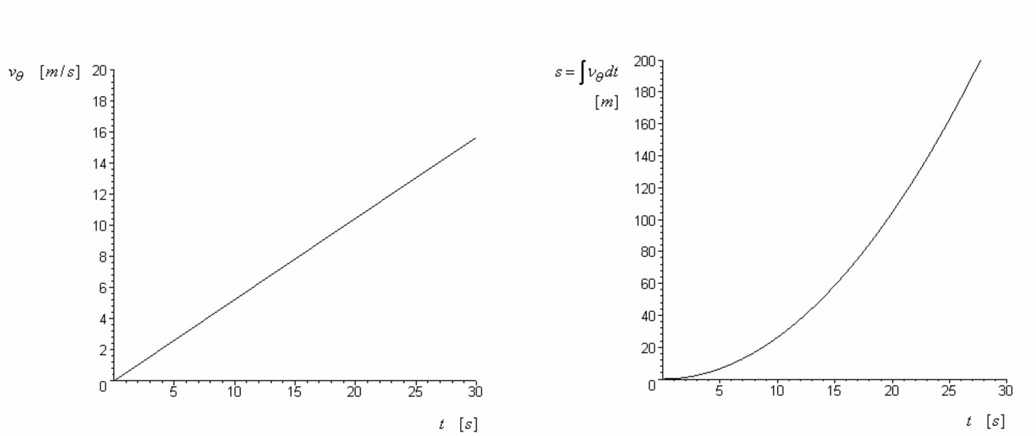

Figure 4-4: Evolution of the slide velocity and slide displacement

5. Appendix I: Analytic expression for the Integral $J_r(\theta_1,\theta_2)$, eq. (2.23)

$$
\begin{aligned}
Jr(\alpha) := -\frac{1}{2}\Big(& -\sin(\theta1)\sin(\alpha)\cos(\theta2)^2+\cos(\theta1)\cos(\alpha)\cos(\theta2)^2+\cos(\theta1)\sin(\alpha)\sin(\theta2)\cos(\theta2) \\
& +\sin(\theta1)\cos(\alpha)\sin(\theta2)\cos(\theta2)-\sin(\alpha)\cos(\alpha)^2\sin(\theta1)-2\cos(\alpha)^3\cos(\theta1)^2\cos(\theta2) \\
& +\sin(\alpha)\cos(\alpha)^2\sin(\theta2)+2\cos(\alpha)^4\,\mathrm{arctanh}((\cos(\theta1)\cos(\alpha)-\cos(\theta1)\cos(\alpha)\cos(\theta2) \\
& -\sin(\theta1)\sin(\alpha)+\sin(\theta1)\sin(\alpha)\cos(\theta2)-\sin(\theta1)\cos(\alpha)\sin(\theta2)-\cos(\theta1)\sin(\alpha)\sin(\theta2))/ \\
& \sin(\theta2))\sin(\theta2)+2\cos(\alpha)^4\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\sin(\theta2) \\
& -\sin(\alpha)\sin(\theta2)\cos(\theta1)^2-2\cos(\alpha)^3\cos(\theta1)\sin(\theta2)\sin(\theta1) \\
& -2\sin(\alpha)\cos(\alpha)^2\cos(\theta1)^2\sin(\theta2)+2\cos(\alpha)^2\,\mathrm{arctanh}((\cos(\theta1)\cos(\alpha) \\
& -\cos(\theta1)\cos(\alpha)\cos(\theta2)-\sin(\theta1)\sin(\alpha)+\sin(\theta1)\sin(\alpha)\cos(\theta2)-\sin(\theta1)\cos(\alpha)\sin(\theta2) \\
& -\cos(\theta1)\sin(\alpha)\sin(\theta2))/\sin(\theta2))\cos(\theta1)^2\sin(\theta2)-2\sin(\alpha)\cos(\alpha)^3\,\mathrm{arctanh}((\cos(\theta1)\cos(\alpha) \\
& -\cos(\theta1)\cos(\alpha)\cos(\theta2)-\sin(\theta1)\sin(\alpha)+\sin(\theta1)\sin(\alpha)\cos(\theta2)-\sin(\theta1)\cos(\alpha)\sin(\theta2) \\
& -\cos(\theta1)\sin(\alpha)\sin(\theta2))/\sin(\theta2))\cos(\theta2) \\
& -2\sin(\alpha)\cos(\alpha)^3\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta2) \\
& +2\cos(\alpha)^2\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta1)^2\sin(\theta2) \\
& -4\cos(\alpha)^4\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta1)^2\sin(\theta2)-4\cos(\alpha)^4 \\
& \mathrm{arctanh}((\cos(\theta1)\cos(\alpha)-\cos(\theta1)\cos(\alpha)\cos(\theta2)-\sin(\theta1)\sin(\alpha)+\sin(\theta1)\sin(\alpha)\cos(\theta2) \\
& -\sin(\theta1)\cos(\alpha)\sin(\theta2)-\cos(\theta1)\sin(\alpha)\sin(\theta2))/\sin(\theta2))\cos(\theta1)^2\sin(\theta2) \\
& +\sin(\alpha)\cos(\theta1)\sin(\theta1)\cos(\theta2)+\cos(\alpha)^3\cos(\theta1)-\cos(\alpha)\cos(\theta2)+\cos(\alpha)^3\cos(\theta2)+4 \\
& \sin(\alpha)\cos(\alpha)^3\,\mathrm{arctanh}((\cos(\theta1)\cos(\alpha)-\cos(\theta1)\cos(\alpha)\cos(\theta2)-\sin(\theta1)\sin(\alpha) \\
& +\sin(\theta1)\sin(\alpha)\cos(\theta2)-\sin(\theta1)\cos(\alpha)\sin(\theta2)-\cos(\theta1)\sin(\alpha)\sin(\theta2))/\sin(\theta2))\cos(\theta1) \\
& \sin(\theta2)\sin(\theta1) \\
& +4\sin(\alpha)\cos(\alpha)^3\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta1)^2\cos(\theta2)+4 \\
& \sin(\alpha)\cos(\alpha)^3\,\mathrm{arctanh}((\cos(\theta1)\cos(\alpha)-\cos(\theta1)\cos(\alpha)\cos(\theta2)-\sin(\theta1)\sin(\alpha) \\
& +\sin(\theta1)\sin(\alpha)\cos(\theta2)-\sin(\theta1)\cos(\alpha)\sin(\theta2)-\cos(\theta1)\sin(\alpha)\sin(\theta2))/\sin(\theta2))\cos(\theta2) \\
& \cos(\theta1)^2+2\sin(\alpha)\cos(\theta1)\cos(\alpha)^2\sin(\theta1)\cos(\theta2) \\
& -2\sin(\theta1)\cos(\alpha)^2\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta1)\cos(\theta2) \\
& +4\sin(\theta1)\cos(\alpha)^4\,\mathrm{arctanh}\left(\frac{-\cos(\theta1)\cos(\alpha)+\sin(\theta1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta1)}\right)\cos(\theta1)\cos(\theta2)-2
\end{aligned}
$$

$$\sin(\theta 1)\cos(\alpha)^2 \operatorname{arctanh}((\cos(\theta 1)\cos(\alpha)-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)-\sin(\theta 1)\sin(\alpha)$$
$$+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)-\sin(\theta 1)\cos(\alpha)\sin(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\theta 1)$$
$$\cos(\theta 2)+4\sin(\theta 1)\cos(\alpha)^4 \operatorname{arctanh}((\cos(\theta 1)\cos(\alpha)-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)-\sin(\theta 1)\sin(\alpha)$$
$$+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)-\sin(\theta 1)\cos(\alpha)\sin(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\theta 1)$$
$$\cos(\theta 2)$$
$$+4\sin(\theta 1)\sin(\alpha)\cos(\alpha)^3 \operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha)+\sin(\theta 1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta 1)}\right)\cos(\theta 1)\sin(\theta 2)\Bigg)$$
$$/(-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2)$$
$$-\sin(\theta 1)\cos(\alpha)\sin(\theta 2))$$

6. Appendix II: Analytic expression for the Integral $J_{\theta}(\theta_1,\theta_2)$, eq. (3.13)

$$
\begin{aligned}
Jt(\alpha) := -\frac{1}{2}\Big(& -\cos(\theta 1)\sin(\alpha) - \sin(\theta 1)\cos(\alpha) + 2\sin(\alpha)\cos(\alpha)^2\cos(\theta 1)\sin(\theta 1)\sin(\theta 2) - 4\sin(\alpha) \\
& \operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) \\
& - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^3\cos(\theta 1)\sin(\theta 1)\cos(\theta 2) \\
& + 4\sin(\alpha)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^3\sin(\theta 2)\cos(\theta 1)^2 + 4 \\
& \operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) \\
& - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^4\cos(\theta 1)\sin(\theta 1)\sin(\theta 2) \\
& + 4\sin(\theta 1)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^4\cos(\theta 1)\sin(\theta 2) \\
& - 2\sin(\theta 1)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^2\cos(\theta 1)\sin(\theta 2) + 4 \\
& \sin(\alpha)\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) \\
& + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^3 \\
& \cos(\theta 1)^2\sin(\theta 2) \\
& - 4\sin(\alpha)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^3\cos(\theta 1)\sin(\theta 1)\cos(\theta 2) \\
& - 2\sin(\theta 1)\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) \\
& + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^2 \\
& \cos(\theta 1)\sin(\theta 2) + \sin(\alpha)\cos(\theta 1)^2\cos(\theta 2) + \sin(\alpha)\cos(\theta 1)\cos(\theta 2)^2 + 2\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) \\
& - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) \\
& - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^2\cos(\theta 2) - 2\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha) \\
& - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) \\
& - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^4\cos(\theta 2) - \sin(\alpha)\cos(\alpha)^2\cos(\theta 2) \\
& + 2\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^2\cos(\theta 2) \\
& - 2\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^4\cos(\theta 2) \\
& - \sin(\alpha)\cos(\theta 1)\cos(\alpha)^2 - 2\cos(\alpha)^3\sin(\theta 2)\cos(\theta 1)^2 + \cos(\alpha)\sin(\theta 1)\cos(\theta 2)^2 \\
& + \sin(\theta 2)\cos(\alpha) + \cos(\alpha)^3\sin(\theta 2) - \cos(\alpha)^3\sin(\theta 1) - \sin(\alpha)\cos(\theta 2) \\
& - \cos(\alpha)\sin(\theta 2)\cos(\theta 1)\cos(\theta 2) + 2\sin(\theta 1)\cos(\theta 1)\cos(\alpha)^3\cos(\theta 2) \\
& - 2\sin(\alpha)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha) + \sin(\theta 1)\sin(\alpha) + \cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^3\sin(\theta 2) + 4\operatorname{arctanh}((\\
& \cos(\theta 1)\cos(\alpha) - \cos(\theta 1)\cos(\alpha)\cos(\theta 2) - \sin(\theta 1)\sin(\alpha) + \sin(\theta 1)\sin(\alpha)\cos(\theta 2) \\
& - \sin(\theta 1)\cos(\alpha)\sin(\theta 2) - \cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^4\cos(\theta 2)\cos(\theta 1)^2 - 2\sin(\alpha)
\end{aligned}
$$

$$\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha)-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)-\sin(\theta 1)\sin(\alpha)+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)$$

$$-\sin(\theta 1)\cos(\alpha)\sin(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^3\sin(\theta 2)$$

$$+4\cos(\alpha)^4\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha)+\sin(\theta 1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta 1)}\right)\cos(\theta 1)^2\cos(\theta 2)$$

$$+2\sin(\alpha)\cos(\theta 1)^2\cos(\alpha)^2\cos(\theta 2)$$

$$-2\cos(\theta 2)\operatorname{arctanh}\left(\frac{-\cos(\theta 1)\cos(\alpha)+\sin(\theta 1)\sin(\alpha)+\cos(\alpha)}{\sin(\theta 1)}\right)\cos(\alpha)^2\cos(\theta 1)^2-2\cos(\theta 2)$$

$$\operatorname{arctanh}((\cos(\theta 1)\cos(\alpha)-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)-\sin(\theta 1)\sin(\alpha)+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)$$

$$-\sin(\theta 1)\cos(\alpha)\sin(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2))/\sin(\theta 2))\cos(\alpha)^2\cos(\theta 1)^2$$

$$\left.+\sin(\alpha)\cos(\theta 1)\sin(\theta 1)\sin(\theta 2)+\sin(\theta 1)\sin(\alpha)\sin(\theta 2)\cos(\theta 2)\right)/(-\cos(\theta 1)\cos(\alpha)\cos(\theta 2)$$

$$+\sin(\theta 1)\sin(\alpha)\cos(\theta 2)-\cos(\theta 1)\sin(\alpha)\sin(\theta 2)-\sin(\theta 1)\cos(\alpha)\sin(\theta 2))$$

7. References

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