The micromechanical definition of the intergranular stress[1]

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1. Definition of an average continuum stress

For the transition from the discrete medium to the continuum we observe that the Cauchy stress tensor obeys equilibrium conditions such that (in the static case) its divergency equilibrates the body forces,

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_i = 0 \tag{1.1}$$

which hold for any point inside a representative elementary volume (REV), i.e. $\forall x_k \in V_{REV}$. On the boundary of the considered (REV) the stress tensor relates to the surface tractions according to the equations of equilibrium

$$\sigma_{ij} n_i = t_j \quad \forall x_k \in \partial V_{REV} \tag{1.2}$$

We define now the mean stress inside the (REV),

$$\bar{\sigma}_{ij} = \frac{1}{V_{REV}} \int_{V_{REV}} \sigma_{ij} dV \tag{1.3}$$

Let,

$$\Pi_{ikj} = x_i \sigma_{kj} \tag{1.4}$$

then,

$$\frac{\partial}{\partial x_k} \Pi_{ikj} = \frac{\partial}{\partial x_k}\left(x_i \sigma_{kj}\right) = \delta_{ki}\sigma_{kj} + x_i \frac{\partial \sigma_{kj}}{\partial x_k} = \sigma_{ij} - x_i f_j \tag{1.5}$$

where eq . (1.1) was used. Thus eq. (1.3) yields,

$$\begin{aligned} \bar{\sigma}_{ij} &= \frac{1}{V_{REV}} \int_{V_{REV}} \left(\frac{\partial}{\partial x_k}\left(x_i \sigma_{kj}\right) + x_i f_j \right) \\ &= \frac{1}{V_{REV}} \int_{\partial V_{REV}} x_i \sigma_{kj} n_k dS + \frac{1}{V_{REV}} \int_{V_{REV}} x_i f dV \\ &= \frac{1}{V_{REV}} \int_{\partial V_{REV}} x_i t_j dS + \frac{1}{V_{REV}} \int_{V_{REV}} x_i f dV \end{aligned} \tag{1.6}$$

where eq. (1.2) was used.

[1] Translated from I. Vardoulakis, *Lecture notes on Continuum Mechanics*, 2006

We observe that if L is the characteristic radius of the REV then the first term on r.h.s. of eq. (1.6) is of the order of L^{-1} whereas the second term is of the order of 1. Thus we may neglect the influence of the body forces in the computation of the average stress. In conclusion we write that

$$\bar{\sigma}_{ij} \approx \frac{1}{V_{REV}} \int_{\partial V_{REV}} x_i t_j dS \qquad (1.7)$$

2. The virtual work equation as applied to a discrete assembly of grains inide the REV[2]

We consider an (REV) in a granular discrete medium, which contains N grains, which may be in contact with each other. Some of the grains on the boundary of the (REV) are receiving forces from grains lying outside the considered (REV). As demonstrated in the continuum case for small (REV) the action body forces is considered as being negligible.

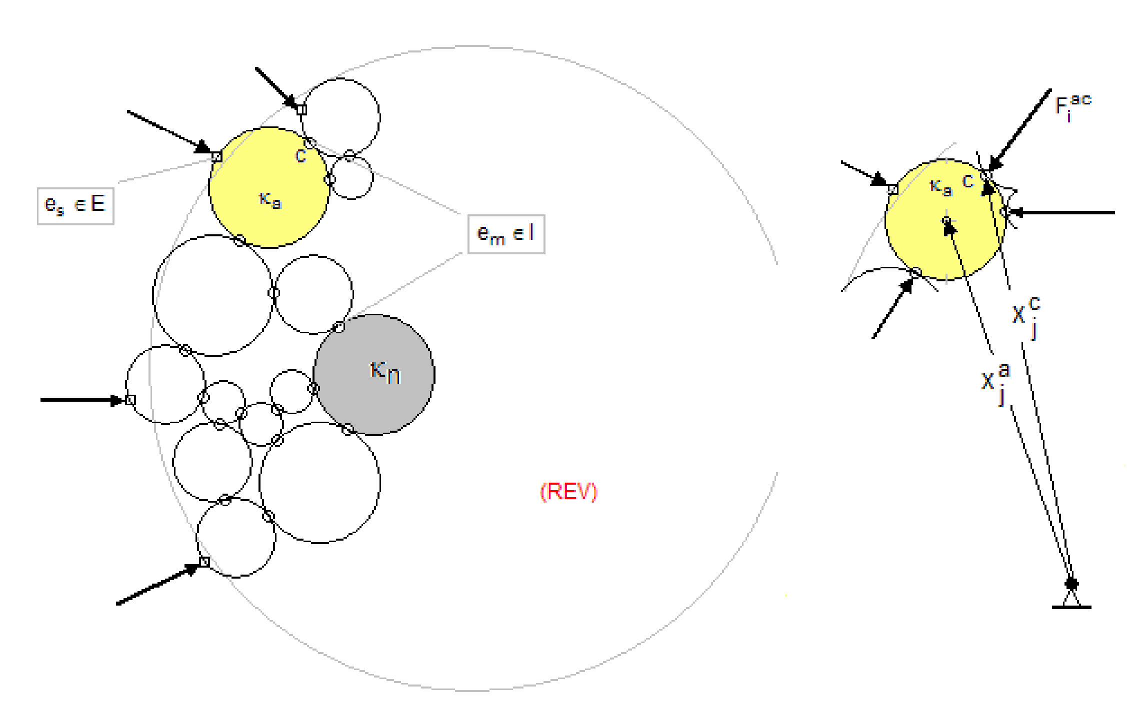

Fig. 1: The (REV) *in a discrete assembly of grains*

[2] Bardet, J.-P. and Vardoulakis (2001). The asymmetry of stress in granular media. *Int. J. Solids Struct.*,38, 353-367.

All grains inside the (REV) are grouped in an index-catalogue which corresponds to the set of indices of their enumeration,

$$B = \{\kappa_1, \ldots, \kappa_a, \ldots, \kappa_N\} \leftrightarrow B = \{1, \ldots, a, \ldots N\} \tag{2.1}$$

The grains are in contact over small areas ΔS_c. Intergranular forces are acting at points c on these contact areas.

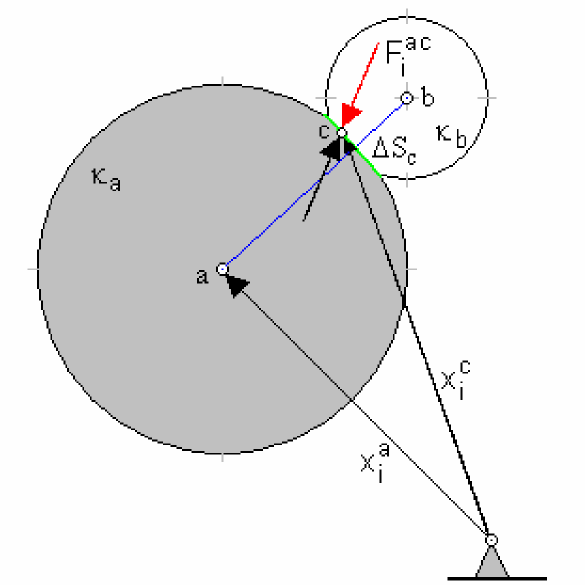

Fig. 2: Two grain contact

These forces are acting on M contact points

$$C = \{c\} = \{e_1, \ldots, e_s, \ldots, e_M\} \leftrightarrow \Gamma = \{1, \ldots, s, \ldots, M\} \tag{2.2}$$

The subset $I \subset C$ contains all contact points in the interior of the (REV). The subset $E \subset C$ contains all contact points on the boundary of the (REV).

$$I = \{e_1, \ldots, e_{M_I}\} \quad \leftrightarrow I = \{1, \ldots, M_I\}$$

$$E = \{e_{M_I+1}, \ldots e_M\} \leftrightarrow E = \{M_I + 1, \ldots M\} \tag{2.3}$$

$$I \cup E = C \;,\; I \cap E = \varnothing$$

The subsets $I_a \leftrightarrow I_a$ and $E_a \leftrightarrow E_a$ contain the contact points of the grain κ_a with grains in the interior, resp. on the exterior of the (REV). The set $C_a \leftrightarrow \Gamma_a$ contains all the contact points of the grain κ_a,

$$C = \bigcup_{a \in B} C_a \ , \ C_a = I_a \cup E_a$$
$$I = \bigcup_{a \in B} I_a \ , \ E = \bigcup_{a \in B} E_a \tag{2.4}$$

For two distinct grains κ_a and κ_b we have,

$$E_a \cap E_b = \varnothing \ , \ I_a \cap \ I_b = \{e_c\} \quad \forall \kappa_a \neq \kappa_b \in B \tag{2.5}$$

A given grain packing is in equilibrium, if every grain is in equilibrium; i.e. if all forces acting on a grain are in equilibrium.

- Force equilibrium:

$$\sum_{c \in C_a} F_i^{ac} = 0 \tag{2.6}$$

- Moment equilibrium:

$$\sum_{c \in C_a} \varepsilon_{ijk} (x_j^c - x_j^a) \ \ F_k^{ac} = 0 \tag{2.7}$$

In these equations x_i^a and x_i^c are the position vectors of the center of gravity and of the contact point, respectively.

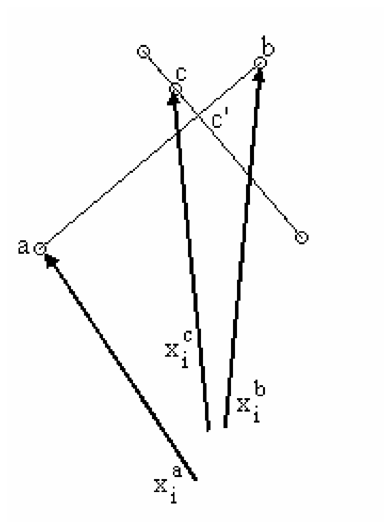

Fig. 3: Position vectors of grain centers and intergranular contact point

We consider a rigid-body virtual displacement of the arbitrary grain κ_a of the assembly, consisting of a virtual displacement δu_i^a and a virtual rotation $\delta\theta_i^a$

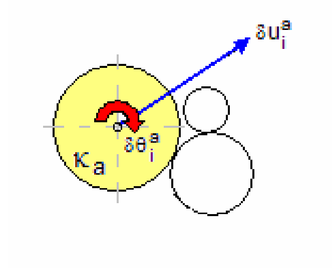

Fig. 4: Rigid body virtual displacement field of a grain

We multiply the above equilibrium conditions, eqs. (2.6) and (2.7) with the virtual displacement- δu_i^a and the virtual rotation field $\delta\theta_i^a$ and we sum them up over all grains inside the considered (REV),

$$\sum_{a\in B}\sum_{c\in C_a}\left(F_i^{ac}\delta u_i^a+\varepsilon_{ijk}\left(x_j^c-x_j^a\right)\ F_k^{ac}\delta\theta_i^a\right)=0 \tag{2.8}$$

This double sum over the sets C_a and B may be regrouped into two sums over the sets I and E. Considering the fact that intergranular forces appear in pairs of opisite forces,

$$F_i^c := F_i^{ac} = -F_i^{bc} \tag{2.9}$$

we get finally the following form of the virtual work equation,

$$\delta W^{(D,ext)} = \delta W^{(D,int)} \tag{2.10}$$

where the quantities $\delta W^{(D,ext)}$ and $\delta W^{(D,int)}$ express respectively[3]:

1. The virtual work of the external forces which act on the discrete medium,

$$\delta W^{(D,ext)} = \sum_{e\in E} F_i^e \delta u_i^e \tag{2.11}$$

where δu_i^e is the virtual displacement of the point of action e of the external load F_i^e.

2. The virtual work of the internal forces which act on the discrete medium:

[3] The index D signifies that these are expressions of virtual work in relation to the discrete medium.

$$\delta W^{(D,int)} = \sum_{c \in I} F_i^c \Delta\delta u_i^c \tag{2.12}$$

where όπου $\Delta\delta u_i^c$ is the displacement at the point of action c of the intergranular force between the grains κ_a and κ_b.

If we neglect the eccentricity of the force resultant and assume that the contact point is on the centerline (ab), then we have

$$\Delta\delta u_i^c = \delta u_i^b - \delta u_i^a + \varepsilon_{ijk}\left(\delta\theta_j^b\left(x_k^c - x_k^b\right) - \delta\theta_j^a\left(x_k^c - x_k^a\right)\right) \tag{2.13}$$

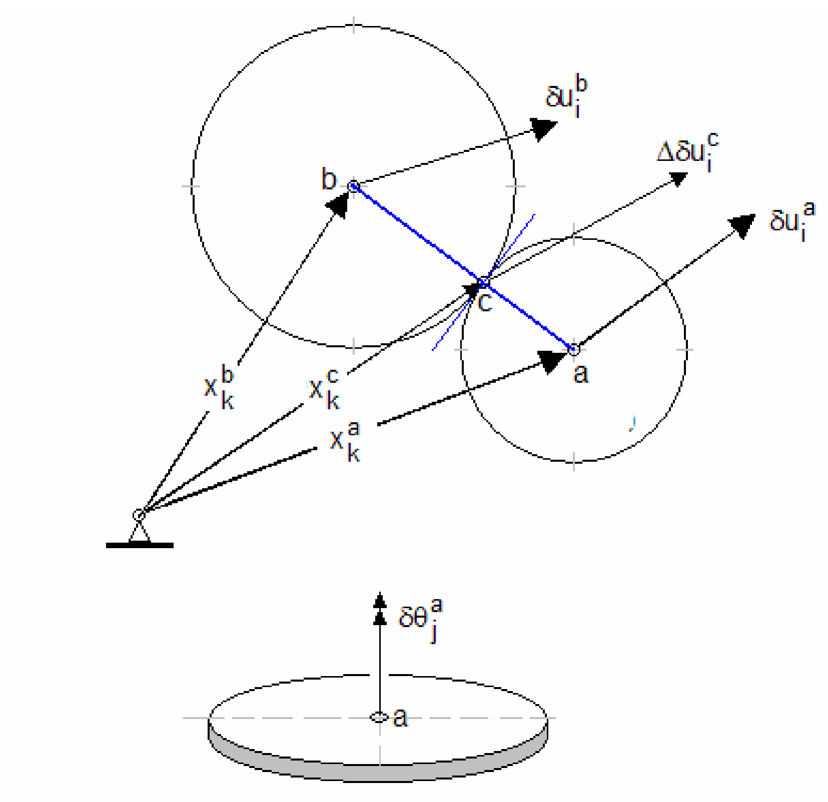

Fig. 5: virtual displacement at the contact point c

The virtual displacement and rotation fields may be chosen arbitrary. In particular they may be chosen as functions of the center point of the grains:

$$\begin{aligned} \delta u_i^a &= a_i + b_{ij} x_j^a + \cdots \\ \delta\theta_i^a &= \alpha_i + \beta_{ij} x_j^a + \cdots \end{aligned} \tag{2.14}$$

with arbitrary coefficients a_i, b_{ij} and α_i, β_{ij}, thus yielding,

$$\begin{aligned}\Delta\delta u_i^c &= b_{ij}\left(x_j^b - x_j^a\right) - \alpha_j\varepsilon_{ijk}\left(x_k^b - x_k^a\right)\\ &+\beta_{jl}\varepsilon_{ijk}\left(x_l^b\left(x_k^c - x_k^b\right) - x_l^a\left(x_k^c - x_k^a\right)\right) + \cdots\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\delta u_i^e &= \delta u_i^a + \varepsilon_{ijk}\delta\theta_j^a\left(x_k^e - x_k^{ae}\right) + \cdots\\ &= a_i + b_{ij}x_j^{ae} + \varepsilon_{ijk}\alpha_j\left(x_k^e - x_k^{ae}\right) + \varepsilon_{ijk}\beta_{jl}x_l^{ae}\left(x_k^e - x_k^{ae}\right) + \cdots\end{aligned} \tag{2.16}$$

where the vector x_k^{ae} connects the center a of the grain κ_a with the external contact point e.

With the above assumptions we get the following expressions for the work of internal and external forces,

$$\delta W^{(D,int)} = b_{ij}\sum_{c\in I} F_i^c(x_j^b - x_j^a) - \alpha_j\sum_{c\in I}\varepsilon_{ijk}F_i^c(x_k^b - x_k^a) + \cdots \tag{2.17}$$

$$\delta W^{(D,ext)} = a_i\sum_{e\in E}F_i^e + b_{ij}\sum_{e\in E}F_i^e x_j^{ae} + \alpha_j\sum_{e\in E}\varepsilon_{ijk}F_i^e(x_k^e - x_k^{ae}) + \cdots \tag{2.18}$$

The virtual work equation (2.10) yields then the following equilibrium equations:

1) The equilibrium of external forces acting on the considered (REV):

$$b_{ij} = 0\ ,\ \alpha_i = 0\ ,\ldots \Rightarrow a_i\sum_{e\in E}F_i^e = 0\quad \forall a_i \Leftrightarrow \sum_{e\in E}F_i^e = 0 \tag{2.19}$$

2) And the conditions,

$$\begin{aligned}&a_i = 0\ ,\ \alpha_i = 0\ ,\ldots \Rightarrow\\ &b_{ij}\sum_{c\in I}F_i^c(x_j^b - x_j^a) = b_{ij}\sum_{e\in E}F_i^e x_j^{ae}\quad \forall b_{ij} \Leftrightarrow\\ &\sum_{c\in I}F_i^c(x_j^b - x_j^a) = \sum_{e\in E}F_i^e x_j^{ae}\end{aligned} \tag{2.20}$$

$$\begin{aligned}&a_i = 0\ ,\ b_{ij} = 0\ ,\ldots \Rightarrow\\ &-\alpha_j\sum_{c\in I}\varepsilon_{ijk}F_i^c(x_k^b - x_k^a) = \alpha_j\sum_{e\in E}\varepsilon_{ijk}F_i^e(x_k^e - x_k^{ae})\quad \forall\alpha_j \Leftrightarrow\\ &-\sum_{c\in I}\varepsilon_{ijk}F_i^c(x_k^b - x_k^a) = \sum_{e\in E}\varepsilon_{ijk}F_i^e(x_k^e - x_k^{ae})\end{aligned} \tag{2.21}$$

If we assume that the quantities $(x_k^e - x_k^{ae})$ are of the order of the grain radius

$$\left| x_k^e - x_k^{ae} \right| = O(R_g) \tag{2.22}$$

then the above equilibrium condition yields approximately to

$$\varepsilon_{ijk} \sum_{c \in I} F_i^c (x_k^b - x_k^a) = 0 \tag{2.23}$$

3. Projection of the discrete medium onto a Boltzmann continuum[4]

We juxtapose eqs.(1.7) and (2.20) we may set,

$$t_j dS \approx F_j^e \tag{3.1}$$

This scheme allows us to compute the mean stress inside the (REV) form the contact forces and the position of the contacts of the grains which are located along the boundary of the (REV)

$$\bar{\sigma}_{ij} \approx \frac{1}{V_{REV}} \sum_{e \in E} x_i^{ae} F_j^e \tag{3.2}$$

which due to eq. (2.20) becomes,

$$\bar{\sigma}_{ij} \approx \frac{1}{V_{REV}} \sum_{c \in I} (x_i^b - x_i^a) F_j^c \tag{3.3}$$

or

$$\bar{\sigma}_{ij} \approx \frac{1}{V_{REV}} \sum_{c \in I} \ell_i^c F_j^c \tag{3.4}$$

where

$$\ell_i = x_i^b - x_i^a \tag{3.5}$$

is the branch vector which connects the centers of the to grains in contact. Eq. (3.5) for the computation of the stress is attributed to Love[5].

We observe finally that from eqs.(3.3) and (2.23) we get that within the considered approximations the mean stress tensor is symmetric.

$$\varepsilon_{ijk} \sum_{c \in I} F_i^c \ell_k^c \approx 0 \Rightarrow \varepsilon_{ijk} \bar{\sigma}_{ki} \approx 0 \tag{3.6}$$

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[4] Although in general flat contacts would provide the ground for the development of couple stresses , these are here not considered.

[5] A.E.H. Love, *A Treatise of the Mathematical Theory of Elasticity*, Cambridge University Press, 1927.

thus for

$$\begin{aligned} & j=1:\ \varepsilon_{312}\bar{\sigma}_{23}+\varepsilon_{213}\bar{\sigma}_{32}\approx 0 \Rightarrow \bar{\sigma}_{23}\approx\bar{\sigma}_{32} \\ & \ldots \\ & \Rightarrow \quad \bar{\sigma}_{ij}\approx\bar{\sigma}_{ji} \end{aligned} \tag{3.7}$$

4. Interparticle stress tensor

Following eq. (3.4) we assume that the bulk soil behavior depends on the interparticle forces. In other words eq. (3.4) relates the suitable continuum stress tensor to the force fabric and it is this tensor that should be used in any constitutive description of the granular medium. The question arises now as of how to determine this constitutive or intergranular stress tensor, using experimental data.

For incompressible grains the presence of an ambient pore-water pressure seems to leave the intergranular forces unaltered. In that case the Terzaghi's effective stress tensor is found **empirically** to be a good measure of the intergranular forces; thus we identify

$$\bar{\sigma}'_{ij}=\frac{1}{V_{REV}}\sum_{c\in I}\ell_i^c F_j^c \tag{4.1}$$

where[6]

$$\bar{\sigma}_{ij}=\bar{\sigma}'_{ij}-p_w\delta_{ij} \tag{4.2}$$

For compressible grains Bishop& Skinner[7] suggested that Terzaghi's concept should be modified in way the interparticle stress should be given by a stress tensor, which deviates from Terzaghi's effective stress tensor by a factor proportional to the pore-pressure and the relative compressibility of the grain c_s to the bulk compressibility c of the (REV); in our terminology this means that

$$\bar{\sigma}_{ij}^{(i)}=\frac{1}{V_{REV}}\sum_{c\in I}\ell_i^c F_j^c \tag{4.3}$$

where

$$\bar{\sigma}_{ij}^{(i)}=\bar{\sigma}'_{ij}-\frac{c_s}{c}p_w\delta_{ij} \tag{4.4}$$

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[6] compression is taken here as negative!

[7] Bishop, A. W. and A.E. Skinner (1977). The influence of high pore-pressure on the strength of cohesionless soils, *Phil. Trans. Roy. Soc.* London, 284, 91-130.

We notice that both eqs. (4.2) and (4.4) are constitutive in nature and can only be tested experimentally.