Chapter 4:
2nd Gradient Flow Theory of Plasticity

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Abstract. This chapter provides a brief introduction to the 2nd Gradient flow theories of plasticity with emphasis on shear-banding.

4.1 Introduction

Localization of deformation leads to a change of scale of the problem, so that phenomena occurring at the scale of the grain cannot be ignored anymore in the mechanical modeling process of the macroscopic behavior of the material. Thus in order to describe correctly localization phenomena it appears necessary to resort to continuum models with micro-structure, such as Cosserat or Gradient models (cf. Vardoulakis & Sulem, 1995). The description of statics and kinematics of continuous media with microstructure has been studied systematically by many authors in the past (cf. Chapter 3). This work was revived in the last two decades of the 20th century, in order to address the problem of localization in Geomaterials.

Here Mindlin's continuum approach is used as the background for the presentation of the most basic results that pertain to more recent works on 2nd gradient elasto-plasticity theories, which can be used for addressing localization phenomena properly.

4.2 A Cohesion-Softening Model for an Ideal Quasi-Brittle Solid

4.2.1 Local Continuum Formulation

As a brief introduction to the problem at hand we describe here a simple mathematical model for quasi brittle solids, by sing a "kinematic hardening" approach (Mroz, 1973, Vardoulakis & Frantziskonis, 1992). Accordingly the Cauchy stress tensor σ_{ij} is split formally into a reduced stress τ_{ij} and into a back-stress α_{ij},

$$\sigma_{ij} = \tau_{ij} + \alpha_{ij} \tag{1}$$

These stress tensors are further decomposed into deviatoric and spherical parts

$$\sigma_{ij} = s_{ij} + p\delta_{ij} \quad , \quad \alpha_{ij} = a_{ij} + q\delta_{ij} \quad , \quad \tau_{ij} = t_{ij} + p_\tau \delta_{ij} \tag{2}$$

For simplicity we assume that plastic yielding is described by a Drucker-Prager yield surface, which is expressed in terms of the reduced stress as follows:

$$F = F(\tau_{ij}) = T_\tau + p_\tau f \tag{3}$$

where p_τ and T_τ are related to the 1st and the 2nd deviatoric invariants of the reduced stress tensor

$$p_\tau = \frac{1}{3} I_{1\tau} = \frac{1}{3}\tau_{kk} \quad , \quad T_\tau = \sqrt{J_{2t}} = \sqrt{\frac{1}{2} t_{ij} t_{ji}} \tag{4}$$

and the "friction" coefficient f is assumed to be constant,

$$f = \text{const.} > 0 \tag{5}$$

We assume also that the back-stress is isotropic,

$$a_{ij} = 0 \quad \Rightarrow \quad \alpha_{ij} = q\delta_{ij} \tag{6}$$

and we get

$$p_\tau = p - q \quad , \quad t_{ij} = s_{ij} \quad \Rightarrow \quad T_\tau \equiv T = \sqrt{J_{2s}} = \sqrt{\frac{1}{2} s_{ij} s_{ji}} \tag{7}$$

With these assumptions Eq. (3) for the yield surface becomes

$$F = T + f\,(p - q) \tag{8}$$

As will be explained below the strength parameter q is assumed to be a decreasing function of the plastic hardening parameter Ψ :

$$q = Q(\Psi) > 0 \downarrow \forall \Psi \geq 0 \Rightarrow q_0 = q(0) > 0 \tag{9}$$

From Fig. 1(a) we see clearly that the evolving yield surface $F(\sigma_{ij}, \Psi) = 0$, Eq. (8), describes the behavior of a constant-friction, cohesion-softening model.

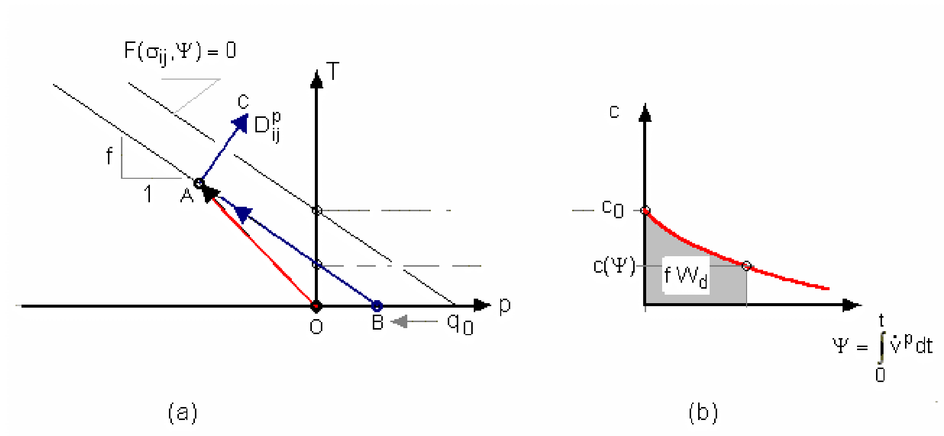

Figure 1. Cohesion softening model

Plastic loading is defined by the conditions

$$F = 0 \quad \Rightarrow \quad T + f\,(p - q) = 0 \tag{10}$$

and

$$\dot{F} = \frac{\partial F}{\partial \tau_{ij}}\,\dot{\tau}_{ij} = 0 \Rightarrow \quad F_{ij}\left(\dot{\sigma}_{ij} - \dot{\alpha}_{ij}\right) = 0 \tag{11}$$

where

$$F_{ij} = \frac{\partial F}{\partial \tau_{ij}} \tag{12}$$

In the considered case of a D.-P. model we have the following simple expressions for the gradient F_{ij}

$$F_{ij} = \frac{1}{\sqrt{2}} m_{ij} + \frac{1}{3} f \delta_{ij} \quad , \quad m_{ij} = \frac{s_{ij}}{\sqrt{2}\,T} \quad , \quad m_{ij} m_{ji} = 1 \tag{13}$$

We assume that the material is dilatant, and we adopt normality as a plastic flow-rule. This assumption may be also seen as a condition for maximum dilatancy,

$$D^p_{ij} = F_{ij}\dot{\Psi} \quad \text{with}: \dot{\Psi} \geq 0 \tag{14}$$

The total dissipation is computed as the work of the stress on the plastic rate of deformation

$$D_{loc} = \sigma_{ij} D_{ij}^{p} = (\tau_{ij} + \alpha_{ij}) D_{ij}^{p} \tag{15}$$

We assume that the work done by the reduced stress on the plastic rate of deformation is dissipated into heat (Mroz, 1973)

$$D_{loc,h} = \tau_{ij} D_{ij}^{p} \tag{16}$$

Due to associated flow-rule, Eq. (14), and the loading condition, Eq. (11), this assumption yields that for the considered model <u>no work</u> is dissipated in heat,

$$D_{loc,h} = \tau_{ij} \dot{\Psi} \frac{\partial F}{\partial \tau_{ij}} = \dot{F} = 0 \tag{17}$$

This can be seen clearly in Fig. 1 (a), where the reduced-stress vector, depicted there by the vector $\vec{\tau} = (\overrightarrow{BA})$, is normal to the plastic rate of deformation, depicted by the vector $\vec{D}^{p} = (\overrightarrow{AB})$. This means in turn that the considered elasto-plastic model describes an ideal quasi-brittle material, for which all dissipation is due to "damage" only

$$D_{loc} = D_{loc,d} = \alpha_{ij} D_{ij}^{p} \tag{18}$$

To this "cold"-type plasticity theory is given sometimes the name "material damage" theory, meaning that energy dissipated during loading is spend mainly for internally damaging the material by e.g. the generation of micro-cracks. Material damage is indirectly observable, since it is usually accompanied by a clear and localizable acoustic emission. Within the mathematical frame of the present constitutive model material damage is observed macroscopically as cohesion softening (Fig. 1 b).

In order to explore further mathematically these assumptions we remark first

$$F_{ij} = \frac{\partial F}{\partial \tau_{ij}} = \frac{\partial F}{\partial \sigma_{kl}} \frac{\partial \sigma_{kl}}{\partial \tau_{ij}} = \frac{\partial F}{\partial \sigma_{ij}} \tag{19}$$

With that from the flow rule, Eq. (14) it follows that the tensors σ_{ij} and D_{ij}^{p} are co-axial. Thus we may apply Maxwell's energy decomposition on the expression for the dissipation, by splitting into a volumetric and a deviatoric part

$$D_{loc} = \sigma_{ij} D_{ij}^{p} = p \dot{v}^{p} + T \dot{g}^{p} \tag{20}$$

In Eq. (20) $\dot{v}^p$ and $\dot{g}^p$ are the plastic volumetric- and shearing strain-rate, respectively,

$$\dot{v}^p = D^p_{kk}\ , \quad \dot{g}^p = \sqrt{2D^{p\prime}_{ij}D^{p\prime}_{ij}}\ , \quad D^p_{ij} = D^{p\prime}_{ij} + \frac{1}{3}D^p_{kk} \tag{21}$$

Due to the assumed normality flow-rule we have that the ratio of plastic volumetric- to plastic shear strain-rate is given by the friction coefficient

$$\frac{\dot{v}^p}{\dot{g}^p} = f = \text{const.} \tag{22}$$

Using the loading condition, Eq. (10) we get from Eq. (20)

$$D_{loc} = \dot{v}^p\left(T\frac{\dot{g}^p}{\dot{v}^p} + p\right) = \dot{v}^p\left(T\frac{1}{f} + p\right) = q\dot{v}^p \tag{23}$$

This result is consistent with the above Eqs. (6) and (18),

$$D_{loc} = q\delta_{ij}D^p_{ij} = \alpha_{ij}D^p_{ij} = D_{loc,d} \quad \Rightarrow \quad D_{loc,d} = q\dot{v}^p \tag{24}$$

Thus, if we select the cumulative plastic volumetric strain as “hardening” parameter,

$$\Psi = \int_0^t \dot{v}^p dt \tag{25}$$

then the strength parameter q can be computed from the cumulative damage energy, as indicated in Fig. 1(b),

$$W_d = \int_0^t D_{loc,d}\,dt = \int_0^{v^p} q(\Psi)\,d\Psi \quad \Leftrightarrow \quad q = \frac{dW_d}{dv^p} \tag{26}$$

The observed dilatancy is describing the porosity increase of the material due to micro-cracking. To illustrate this statement we recall the rate equation between the plastic volumetric strain and the porosity ϕ of the material (cf. Vardoulakis & Sulem, 1995)

$$\dot{v}^p \approx \frac{\dot{\phi}}{1-\phi} \tag{27}$$

In Eq. (27) elastic volumetric strains are neglected.

These observations allow us to assume that the strength parameter q should be seen in general as a decreasing function of the porosity of the material; i.e.

$$q = Q(\phi) \tag{28}$$

with

$$H_t = \frac{dQ}{d\phi} < 0 \tag{29}$$

Remark

The simplest model of strength reduction due to porosity increase a “damage” power law of the form,

$$q = q_0 (1 - \phi)^n$$

resulting to

$$H_t = -n q_0 (1 - \phi)^{n-1}$$

For $n = 1$ we get a Kachanov-type model with constant rate of softening,

$$q = q_0 (1 - \phi) \quad \Rightarrow \quad H_t = -q_0$$

The major defect of such constitutive models remains the fact that these are strain-softening models. The application of such a softening models leads to mathematically ill-posed boundary-value problem formulations. This mathematical ill-posedeness can be easily verified by means of a linear stability analysis. Such an analysis shows that in the limit a perturbation of vanishing wave-length grows with infinite pace (Schaeffer, 1990). This theoretical result indicates that damage in the softening regime has the tendency to localize in a narrow zone, which macroscopically is interpreted as a “fracture” surface. The classical continuum approach, pursued up to this point is not capable to simulate the localization of the deformation in a finite strip, leading thus to zero-fracture-zone thickness and to infinite Ljapunov exponents for the time evolution of the instability.

4.2.2 The Non-Local Assumption

A remedy for the defect of the classical continuum with strain softening is to modify the constitutive Eq. (28). Accordingly we replace this “local” continuum assumption by a non-local one (Vardoulakis & Aifantis, 1989): Through Eq. (28) we assumed in fact that the strength is a function of the local value of the porosity. Accordingly we reformulate the above statement by stating instead that the strength is a function of the mean value of the porosity over a small but finite characteristic volume, V_c, whose center of gravity coincides with the position of the considered material point,

$$q = Q(< \phi >) \tag{30}$$

$$< \phi >= \frac{1}{V_c} \int_{V_c} \phi \, dV \tag{31}$$

For the evaluation of the integral in Eq. (31) we put the origin of the co-ordinate system at the considered material point. Let $x_i = h_i$ be the coordinates of an arbitrary point inside the volume of integration V_c. By using a truncated Taylor series expansion we get,

$$\phi(x_i) = \phi(0) + \left.\frac{\partial \phi}{\partial x_i}\right|_0 h_i + \frac{1}{2} \left.\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right|_0 h_i h_j + O\left(h_i^3\right) \tag{32}$$

We assume that V_c is a cube, whose sides have the length $\bar{\ell}_c$.

From Eqs. (31) and (32) we get the following well-known integration rule from Numerical Analysis[1],

$$< \phi > \approx \phi + \frac{\bar{\ell}_c^2}{24} \nabla^2 \phi \tag{33}$$

We observe that “local homogeneity” means that in Eq. (33) the term with the Lapalcian is negligible, thus yielding the classical result

$$< \phi > = \phi \tag{34}$$

Thus in a locally homogeneous setting the value of a field at point coincides with its mean value over an elementary volume that is centered at this point. Eq. (34) is equivalent with a linear-in-space variation of the field $\phi(x_i)$. In a locally non-homogeneous setting, Eq. (33) is used instead, for the approximation of the mean value over the sampling volume. In this approximation we assumed that the characteristic length

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[1] cf. Henrici, P. *Elements of Numerical Analysis*, Wiley, 1964.

$$\ell_c = \bar{\ell}_c / (2\sqrt{6}) \tag{35}$$

is a small parameter, as compared to any other geometric length of the problem at hand.

With these assumptions the non-local constitutive Eq. (30) yields

$$q \approx Q\left(\phi + \ell_c^2 \nabla^2 \phi\right) \approx Q(\phi) + \frac{dQ}{d\phi} \ell_c^2 \nabla^2 \phi \tag{36}$$

or with Eq. (29) to the following expression,

$$q \approx Q(\phi) - \ell_c^2 |H_t| \nabla^2 \phi \quad (H_t < 0) \tag{37}$$

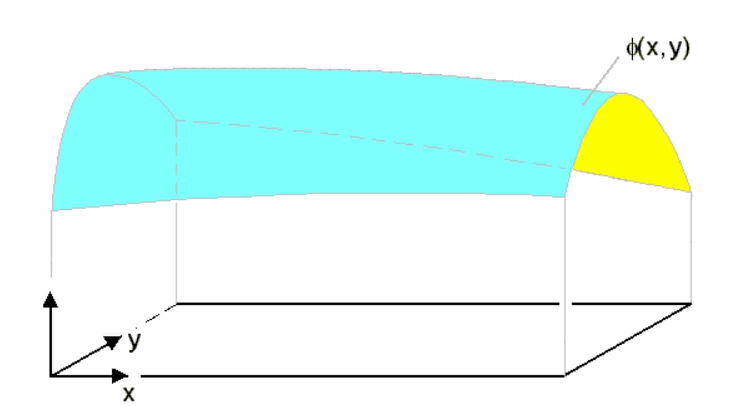

Figure 2. Negative mean curvature of the porosity distribution function in regions of damage localization

The physical meaning of this non-local modification of the constitutive law for the strength is the following (Fig.2): It is expected that within a zone of damaged material the porosity (due to micro-cracking) assumes large values. This is expressed by a negative “mean-curvature condition”, (or “localization”)

$$\nabla^2 \phi < 0 \tag{38}$$

Thus the non-local term in Eq. (37) acts in a way to counterbalance effect of softening.

From Eq. (37) by material time differentiation we get,

$$\dot{q} \approx \left(H_t + \frac{dH_t}{d\phi} \ell_c^2 \nabla^2 \phi \right) \dot{\phi} + H_t \ell_c^2 \nabla^2 \dot{\phi} \approx H_t \left(\dot{\phi} + \ell_c^2 \nabla^2 \dot{\phi} \right) \tag{39}$$

or due to Eqs. (27) and (25)

$$\overline{H}_t \dot{\Psi} < \dot{q} \approx \overline{H}_t \left(\dot{\Psi} + \ell_c^2 \nabla^2 \dot{\Psi}\right) < 0 \tag{40}$$

where

$$\overline{H}_t = (1 - \phi) \frac{dQ}{d\phi} < 0 \tag{41}$$

From the loading condition, Eq. (11), the constitutive equations for the Cauchy stress and the back stress

$$\dot{\sigma}_{ij} = C^e_{ijkl} D^e_{kl} = C^e_{ijkl} (D_{kl} - \dot{\Psi} F_{kl}) \tag{42}$$

$$\dot{\alpha}_{ij} = \dot{q} \delta_{ij} \tag{43}$$

and the evolution law for the strength parameter q, Eq. (40) we obtain consistency condition,

$$\begin{aligned}
&\dot{F} = F_{ij} \dot{\tau}_{ij} = 0 \Rightarrow \quad F_{ij}\left(\dot{\sigma}_{ij} - \dot{\alpha}_{ij}\right) = 0 \\
&F_{ij} C^e_{ijkl} D_{kl} - F_{ij} C^e_{ijkl} F_{kl} \dot{\Psi} - F_{ij} \overline{H}_t (\dot{\Psi} + \ell_c^2 \nabla^2 \dot{\Psi}) \delta_{ij} = 0 \\
&B_{kl} D_{kl} = H \dot{\Psi} + F_{kk} \overline{H}_t \ell_c^2 \nabla^2 \dot{\Psi}
\end{aligned}$$

or

$$\dot{\Psi} - \ell^2 \nabla^2 \dot{\Psi} = \frac{1}{H} B_{kl} D_{kl} \tag{44}$$

where

$$B_{kl} = F_{ij} C^e_{ijkl} \tag{45}$$

$$H_0 = B_{kl} F_{kl} \tag{46}$$

$$H = H_0 + F_{kk} \overline{H}_t \quad (\overline{H}_t = -\left|\overline{H}_t\right|) \tag{47}$$

and

$$\ell^2 = \frac{|H_t|}{H} F_{kk} \ell_c^2 \tag{48}$$

We require that the plastic modulus is positive, thus imposing a restriction to the softening modulus not to exceed the snap-back threshold value H_0 (Nguyen & Bui, 1974):

$$H > 0 \quad \Rightarrow \quad F_{kk} |\overline{H}_t| < H_0 \tag{49}$$

With

$$F_{kk} = f \quad , \quad 0 < f < 0.87 \tag{50}$$

and

$$H_0 = G\left(1 + \frac{2}{3}\frac{1+\nu}{1-2\nu} f^2\right) \tag{51}$$

Ineq. (49) is satisfied as soon as the softening rate is not exceeding the elastic shear modulus

$$|H_t| < G \tag{52}$$

Thus the evolving material length-parameter, defined above through Eq. (48), is always real,

$$\ell^2 > 0 \tag{53}$$

For homogeneous ground plastic-strain states, the term with the Laplacian on the l.h.s. of Eq. (44) vanishes identically and the consistency condition collapses to that of the classical flow theory of plasticity; i.e. to a monomial equation, which can be readily solved in terms of the plastic multiplier $\dot{\Psi}$. In general direct elimination of $\dot{\Psi}$ will not be possible and one has to carry the consistency condition as an additional field equation together with the balance equations, and to treat $\dot{\Psi}$ as an additional degree of freedom (Mühlhaus & Aifantis,1991)[7]. An approximate procedure however proposed by Vardoulakis et al. (1989 and 1992) allows for the elimination of $\dot{\Psi}$ from the set of governing equations and simplifies the problem drastically (see Appendix I),

$$\dot{\Psi} = \frac{1}{1-\ell^2\nabla^2}\left(\frac{1}{H} B_{kl} D_{kl}\right) \quad \Rightarrow \quad \dot{\Psi} \approx \frac{<1>}{H} B_{kl}\left(1 + \ell^2\nabla^2\right) D_{kl} \tag{54}$$

where $< \cdot >$ denote the Föppl-Macauley brackets, so that $\dot{\Psi} \geq 0$; i.e. with

$$H > 0 \tag{55}$$

we define the switch function:

$$<1>=\begin{cases}1 & \text{if}: F=0 \text{ and } B_{kl}\left(1+\ell^2\nabla^2\right)D_{kl}>0\\ 0 & \text{if}: F<0 \text{ or } F=0 \text{ and } B_{kl}\left(1+\ell^2\nabla^2\right)D_{kl}\leq 0\end{cases} \tag{56}$$

Above approximation, Eq. (54), allows to derive explicitly the rate constitutive equations of gradient flow theory of plasticity. Indeed, from Eqs. (42) and (54) we get

$$\dot{\sigma}_{ij}=C^{e}_{ijkl}\,D^{e}_{kl}\quad\Rightarrow$$

$$\dot{\sigma}_{ij}=C^{e}_{ijkl}D_{kl}-C^{e}_{ijmn}F_{mn}\frac{<1>}{H}B_{kl}\left(1+\ell^2\nabla^2\right)D_{kl}$$
$$=C^{e}_{ijkl}D_{kl}-\frac{<1>}{H}C^{e}_{ijmn}F_{mn}B_{kl}D_{kl}-\frac{<1>}{H}C^{e}_{ijmn}F_{mn}B_{kl}B_{kl}\ell^2\nabla^2\,D_{kl}$$

or

$$\dot{\sigma}_{ij}=C^{ep}_{ijkl}\,D_{kl}-C^{p}_{ijkl}\,\ell^2\nabla^2 D_{kl} \tag{57}$$

where

$$C^{ep}_{ijkl}=C^{e}_{ijkl}-C^{p}_{ijkl}\quad,\quad C^{p}_{ijkl}=\frac{<1>}{H}C^{e}_{ijmn}F_{mn}F_{pq}C^{e}_{pqkl} \tag{58}$$

Notice that the constitutive Eqs. (57) constitute a singular perturbation of the ones of classical flow theory.

4.3 Isotropic Softening Plasticity

The previous considerations allow us to produce the following approach to a 2nd gradient plasticity model for isotropically softening solid. We start with a generalization of the yield function, by assuming that it is of the form,

$$F=\hat{F}(\sigma_{ij},<\Psi>) \tag{59}$$

where

$$< \Psi >= \frac{1}{V_c} \int_{V_c} \Psi dV \tag{60}$$

is the average of the hardening parameter over the characteristic volume V_c. For example in Critical State Soil Plasticity Theory we assume that the yield function depends on the current state of the voids ratio, $\Psi = e$. In that case Eq. (59) would reflect the fact that porosity is to be evaluated over a characteristic volume V_c, and that the assumption $< e >= e$ simply reflects a hypothesis about the local homogeneity of the porosity distribution. With

$$< \Psi > \approx \Psi + \ell_c^2 \nabla^2 \Psi \tag{61}$$

we get from Eq. (59),

$$F \approx \hat{F}(\sigma_{ij} \Psi) + \frac{\partial \hat{F}}{\partial \Psi} \ell_c^2 \nabla^2 \Psi \tag{62}$$

For a softening solid the hardening modulus is negative,

$$H_t = -\frac{\partial \hat{F}}{\partial \Psi} < 0 \quad \Rightarrow \quad |H_t| = -H_t = \frac{\partial \hat{F}}{\partial \Psi} > 0 \tag{63}$$

and Eq. (62) becomes,

$$F \approx \hat{F}(\sigma_{ij} \Psi) + |H_t| \ell_c^2 \nabla^2 \Psi \tag{64}$$

This yield function is now used together with constitutive equations of elasto-plasticity theory,

$$D_{ij} = D_{ij}^e + D_{ij}^p \tag{65.1}$$

$$D_{ij}^e = C_{ijkl}^{e\,-1} \, \dot{\sigma}_{kl} \tag{65.2}$$

$$D_{ij}^p = \dot{\Psi} \frac{\partial Q}{\partial \sigma_{ij}} \quad , \quad \dot{\Psi} \geq 0 \tag{65.3}$$

such that plastic strains are generated whenever the stress is and stays on the current yield surface.

With these remarks the consistency condition becomes,

$$\dot{F} = 0 \quad \Rightarrow \quad F \approx \dot{\hat{F}}(\sigma_{ij} \Psi) + |H_t| \ell_c^2 \nabla^2 \dot{\Psi} + O\left(\nabla^2 \Psi\right) = 0$$

or

$$\frac{\partial \hat{F}}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial \hat{F}}{\partial \Psi} \dot{\Psi} + \left| H_t \right| \ell_c^2 \nabla^2 \dot{\Psi} + O\left(\nabla^2 \Psi\right) = 0$$

or

$$\frac{\partial F}{\partial \sigma_{ij}} C^e_{ijkl} \left(D_{kl} - \dot{\Psi} \frac{\partial Q}{\partial \sigma_{ij}} \right) - H_t \dot{\Psi} + \left| H_t \right| \ell_c^2 \nabla^2 \dot{\Psi} + O\left(\nabla^2 \Psi\right) = 0$$

or

$$(1 - \ell^2 \nabla^2) \dot{\Psi} \approx \frac{1}{H} B_{kl} D_{kl} \tag{66}$$

where

$$\begin{gathered} F_{ij} = \frac{\partial \hat{F}}{\partial \sigma_{ij}} \quad , \quad Q_{ij} = \frac{\partial Q}{\partial \sigma_{ij}} \\ B_{kl} = F_{ij} C^e_{ijkl} \\ H_0 = F_{ij} C^e_{ijkl} Q_{kl} \\ H = H_0 + H_t > 0 \end{gathered} \tag{67}$$

and

$$\ell^2 = \frac{\left| H_t \right|}{H} \ell_c^2 > 0 \tag{68}$$

Within a good approximation, Eq. (66) can be inverted, yielding thus again the Eq. (54) and the loading condition, Eq. (56). The corresponding rate equations of 2nd gradient non-associated elasto-plasticity theory are:

$$\dot{\sigma}_{ij} = C^{ep}_{ijkl} D_{kl} - C^p_{ijkl} \ell^2 \nabla^2 D_{kl} \tag{69}$$

$$C^{ep}_{ijkl} = C^e_{ijkl} - C^p_{ijkl} \quad , \quad C^p_{ijkl} = \frac{<1>}{H} C^e_{ijmn} F_{mn} Q_{pq} C^e_{pqkl} \tag{70}$$

4.4 The Mindlin-Continuum Formulation

The constitutive Eqs. (57) or (69) are the rate-equations of a 2nd gradient elasto-plastic solid. In view of these constitutive equations we introduce decomposition of the Cauchy stress rate

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}^{(0)} + \dot{\sigma}_{ij}^{(2)} \tag{71}$$

The first term on the r.h.s. of Eq. (71) is called the 0th grade constitutive stress-rate and set equal to the constitutive stress-rate of the classical continuum

$$\dot{\sigma}_{ij}^{(0)} = C_{ijkl}^{ep}\, D_{kl} \tag{72}$$

The second term is called the 2nd grade constitutive stress-rate and it is assumed to satisfy an internal balance equation,

$$\dot{\sigma}_{ij}^{(2)} + \partial_k \dot{m}_{kij} = 0 \tag{73}$$

where $\dot{m}_{ijk}$ has the dimensions of a double-stress-rate and is defined through the following constitutive equation of gradient type,

$$\dot{m}_{ijk} = C_{jkmn}^{p} \ell^2 \partial_i D_{mn} \tag{74}$$

We see that, within a reasonable approximation, from Eqs. (71) to (74) we recover, the original constitutive Eqs. (57) or (69) of 2nd Gradient elasto-plasticity.

We consider a virtual kinematical set that is derived from a virtual velocity field δv_i,

$$\delta D_{ij} = \frac{1}{2}\left(\partial_i \delta v_j + \partial_j \delta v_i\right) \quad , \quad \delta c_{ijk} = \partial_i \delta D_{jk} \tag{75}$$

We assume that the stress-rate tensor $\dot{\sigma}_{ij}^{(0)}$ is working on the rate of deformation tensor and the double stress-rate tensor $\dot{m}_{ijk}$ is working on the gradient of the rate of deformation tensor. Accordingly we define the total second-order virtual power of internal forces as follows (cf. Chapter 3, Eqs. (7.2) and (35)):

$$\delta_2 P^{(i)} = \int_V \left(\dot{\sigma}_{ij}^{(0)} \delta D_{ij} + \dot{m}_{ijk} \delta c_{ijk}\right) dV \tag{76}$$

In case of locally homogeneous deformations, the second term inside the integral of Eq. (76) is vanishing, and the stress tensor $\dot{\sigma}_{ij}^{(0)}$ collapses to the Cauchy stress-rate tensor ($\dot{\sigma}_{ij} = \dot{\sigma}_{ij}^{(0)}$). In this case the expression for the 2nd order stress power collapses also to the one holding for the classical continuum.

In a similar way we define the virtual power of the rates of the external forces (cf. Chapter 3, Eq. (62))

$$\delta_2 P^{(e)} = \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k\right) dS \tag{77}$$

where

$$D = n_k \partial_k \quad , \quad D_k = \partial_k - n_k D \tag{78}$$

denotes the normal-to-the-boundary derivative. As explained in Chapter 3 (Appendix IV), the surface integral in Eq. (77) has the following meaning: If we prescribe at boundary point the velocity component $v_{(i)}$ we have also the freedom to specify independently is normal derivative $Dv_{(i)}$.

We require, that the "virtual work" equation holds as an expression of equilibrium

$$\delta_2 P^{(i)} = \delta_2 P^{(e)}$$

or explicitly,

$$\int_V \left(\dot{\sigma}_{ij}^{(0)} \delta D_{ij} + \dot{m}_{ijk} \partial_i \delta D_{jk}\right) dV = \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k\right) dS \tag{79}$$

Using Gauss' theorem this integral equation becomes,

$$\int_V \left(\dot{\sigma}_{ij}^{(0)} - \partial_k \dot{m}_{kij}\right) \delta D_{ij} dV + \int_{\partial V} n_i \dot{m}_{ijk} \delta D_{jk} dS = \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k\right) dS$$

or

$$\int_V \left(\partial_i \left(\dot{\sigma}_{ij} \delta v_j\right) - \partial_i \dot{\sigma}_{ij} \delta v_j\right) dV + \int_{\partial V} n_i \dot{m}_{ijk} \delta D_{jk} dS = \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k\right) dS$$

or

$$
\begin{aligned}
&\int_{\partial V} n_i \dot{\sigma}_{ij} \delta v_j dS - \int_V \partial_i \dot{\sigma}_{ij} \delta v_j \, dV + \int_{\partial V} n_i \dot{m}_{ijk} \delta D_{jk} dS \\
&= \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k \right) dS
\end{aligned}
\tag{80}
$$

where

$$
\dot{\sigma}_{ij} = \dot{\sigma}^{(0)}_{ij} - \partial_k m_{kij} \tag{81}
$$

In the virtual work Eq. (80) the volume integrals yield that indeed the stress-rate $\dot{\sigma}_{ij}$ is a balance stress-rate tensor,

$$
\partial_i \dot{\sigma}_{ij} + \dot{F}_j = 0 \tag{82}
$$

This result means that the virtual work Eq. (80), is in fact a weak form of equilibrium.

As shown in detail in Mindlin & Eshel (1968) and in Vardoulakis & Sulem (1995) the surface integrals in Eq. (80) yield the boundary conditions for the surface potencies, $\dot{P}_k$ and $\dot{Q}_k$

$$
\dot{\sigma}_{ki} n_{ki} - D_l (\dot{m}_{kli} n_k) + (D_p n_p) \dot{m}_{kli} n_k n_l = \dot{P}_i \tag{83.1}
$$

$$
\dot{m}_{kli} n_k n_l = \dot{Q}_i \tag{83.2}
$$

where the stress rates are given by the constitutive equations (71), (72) and (74).

We notice finally that the virtual Eq. (79) together with the constitutive Eqs. (74) and (75), can be used as the basis of Finite Element formulations for the numerical analysis of localization problems (Zervos et al., 2001 a &b)

$$
\int_V \left(C^{ep}_{ijkl} D_{kl} \delta D_{ij} + C^{p}_{jkmn} \ell^2 \partial_i D_{mn} \partial_i \delta D_{jk} \right) dV = \int_V \dot{F}_k \delta v_k dV + \int_{\partial V} \left(\dot{P}_k \delta v_k + \dot{Q}_k D\delta v_k \right) dS \tag{84}
$$

Remarks

1. As we saw in Chapter 3, the introduction of higher gradients into the constitutive relation necessarily leads to the definition of higher-order boundary conditions. A consistent mathematical model for strain-softening elasto-plastic material should be equipped with an elasticity that possesses the same order in higher gradients as the plasticity (Zervos et al. 2001 a & b). This is because one does not know before hand where unloading will take place, and boundary conditions should be defined everywhere to be of the same order.

This remark results in a modification of previous 2nd gradient elasto-plastic models, since we recognize that consistent to gradient plasticity is a gradient elasticity. Accordingly we should set that the background elasticity is non-local as well

$$\dot{\sigma}_{ij} = C^{e}_{ijkl}(1-\ell_e^2\nabla^2)D^{e}_{ij}$$

where $\ell_e << \ell$ is a suitable elastic material length.

2. As already outlined above, the appearance of strain-rate gradients into the constitutive equations raises naturally the problem of higher order boundary conditions. Since strain-gradients become important in phenomena of strain localization, one realizes that the simplest class of boundary value problems, which involve strain gradients and the effect of higher-order boundary conditions is that of interface localizations. A discussion of such a model problem can be found in Vardoulakis et al. (1992).

4.5 Shear- Banding

We consider a plane-strain problem and for a small strain formulation. In that case the differential equations that govern continued equilibrium from a given configuration $C^{(t)}$ are

$$\begin{aligned} &\partial_1\dot{\sigma}_{11} + \partial_2\dot{\sigma}_{21} = 0 \\ &\partial_1\dot{\sigma}_{12} + \partial_2\dot{\sigma}_{22} = 0 \end{aligned} \tag{86}$$

We assume that in the configuration $C^{(t)}$ the fields of initial stress and hardening parameter vary slowly in space. Accordingly, the components of the plastic stiffness tensor C^{p}_{ijkl} may be treated as constants. For convenience, the governing equations are written in the coordinate system of the principal axes of the stress tensor in $C^{(t)}$. Under fully loading conditions we obtain the following set of rate-constitutive equations

$$\begin{aligned} &\dot{\sigma}_{11} = C^{u}_{1111}\dot{\varepsilon}_{11} + C^{u}_{1122}\dot{\varepsilon}_{22} - \ell^2 C^{p}_{1111}\nabla^2\dot{\varepsilon}_{11} - \ell^2 C^{p}_{1122}\nabla^2\dot{\varepsilon}_{22} \\ &\dot{\sigma}_{22} = C^{u}_{2211}\dot{\varepsilon}_{11} + C^{u}_{2222}\dot{\varepsilon}_{22} - \ell^2 C^{p}_{2211}\nabla^2\dot{\varepsilon}_{11} - \ell^2 C^{p}_{2222}\nabla^2\dot{\varepsilon}_{22} \\ &\dot{\sigma}_{12} = 2G\dot{\varepsilon}_{12} \end{aligned} \tag{87}$$

where the components of the stiffness tensors are defined according to Eq. (58) or (70) for $<1>=1$. Thus with, $C^{u}_{ijkl}=C^{e}_{ijkl}+C^{p}_{ijkl}$ we denote the components of the corresponding upper-bound linear comparison solid[2] (cf. Vardoulakis, 1994).

Introducing these expressions into the equilibrium equations results finally into the following system of partial differential equations for the components of the velocity vector

$$\ell^2 C^{p}_{1111}\nabla^2\left(\partial^2_{11}v_1\right)+\ell^2 C^{p}_{1122}\,\nabla^2\left(\partial^2_{21}v_2\right)-C^{u}_{1111}\partial^2_{11}v_1-G\partial^2_{22}v_1-\left(C^{u}_{1122}+G\right)\partial^2_{12}v_2=0 \tag{88.1}$$

$$\ell^2 C^{p}_{2211}\nabla^2\left(\partial^2_{12}v_1\right)+\ell^2 C^{p}_{2222}\nabla^2\left(\partial^2_{22}v_2\right)-\left(C^{u}_{2211}+G\right)\partial^2_{12}v_1-G\partial^2_{11}v_2-C^{u}_{2222}\partial^2_{22}v_2=0 \tag{88.2}$$

These are 4th order partial differential equations for the velocity field and constitute a singular perturbation of the classical ones, which are of 2nd order.

Above p.d.e's. will be investigated here for the special case where solutions are sought that correspond to the localization of deformation into narrow zones of intense shear, the so-called shear-bands. According to Fig. 3, the (x_1, x_2)-coordinate system is chosen in such a fashion that the x_1-axis coincides with the minor minimum principal stress σ_1 in C.

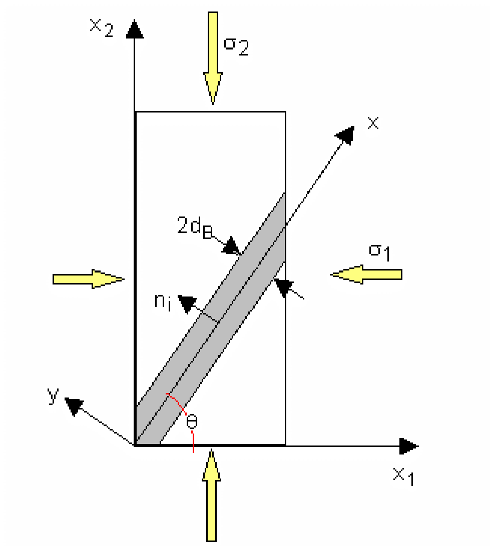

Figure 3. Shear band in an element test

[2] Shanley's or "upper-bound", linear comparison-solid is the solid, which corresponds to continuous loading. We notice that Raniecki & Bruhns (1981) defined a "lower bound", linear comparison-solid as well. These linear comparison solids provide upper and lower bounds for the load (strain) at which shear-band bifurcation is possible.

Let us assume that a shear-band is forming that is inclined with respect to the x_1-axis at an angle θ. A new coordinate system is introduced with its axes parallel and normal to the shear band

$$x = x_1 n_2 - x_2 n_1 \quad ; \quad y = x_1 n_1 + x_2 n_2 \tag{89}$$

where

$$n_1 = -\sin\theta \quad ; \quad n_2 = \cos\theta \tag{90}$$

is the unit vector that is normal to the shear band axis.

By assuming that all field properties related to the forming shear band do not depend on the longitudinal x-coordinate and by setting $(\bullet)' \equiv d/dy$, above Eqs. (88) reduce then to the following system of ordinary differential equations:

$$\begin{aligned} &\ell^2 C^p_{1111} n_1^2 v_1^{(4)} + \ell^2 C^p_{1122} n_1 n_2 v_2^{(4)} - \Gamma_{11} v_1'' - \Gamma_{12} v_2'' = 0 \\ &\ell^2 C^p_{2211} n_1 n_1 v_1^{(4)} + \ell^2 C^p_{2222} n_2^2 v_2^{(4)} - \Gamma_{21} v_1'' - \Gamma_{22} v_2'' = 0 \end{aligned} \tag{91}$$

In Eqs. (91) the tensor Γ_{ik} coincides with the acoustic tensor of the so-called upper-bound linear comparison solid for the direction n_i (cf. Vardoulakis & Sulem, 1995),

$$\Gamma_{ik} = C^u_{ijkl} n_j n_l \tag{92}$$

We search for periodic solutions of the system of Eqs. (79), which have the form (Fig. 4)

$$v_i = -\zeta_i \sin(Qy) \quad (i = 1,2) \tag{93}$$

and fulfill the following boundary conditions,

$$y = \pm d_B : \quad v_i = \mp\zeta_i \tag{94}$$

where $2d_B$ is the shear-band thickness. Thus the "wave number" Q in Eq. (93) is inversely proportional to the shear-band thickness,

$$Q = \frac{\pi}{2d_B} \tag{95}$$

For this velocity field the system of ordinary differential Eqs. (91) reduces to the following algebraic system of equations,

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{96}$$

where

$$\left[b_{ij}\right] = \begin{bmatrix} \ell^2 C^p_{1111} n_1^2 Q^4 + \Gamma_{11} Q^2 & \ell^2 C^p_{1122} n_1 n_2 Q^4 + \Gamma_{12} Q^2 \\ \ell^2 C^p_{2211} n_1 n_2 Q^4 + \Gamma_{21} Q^2 & \ell^2 C^p_{2222} n_2^2 Q^4 + \Gamma_{22} Q^2 \end{bmatrix} \tag{97}$$

For non-trivial solutions in terms of the amplitude vector ζ_i the above system of Eqs. (96) results to the following condition for the shear-band thickness,

$$a_0\left(n_1, n_2\right)\left(Q\ell\right)^2 + a_1\left(n_1, n_2\right) = 0 \tag{98}$$

The coefficients of the above “dispersion equation” are quadratic forms of the components of the unit normal vector

$$\begin{aligned} a_0 &= C^p_{1111}\Gamma_{22} n_1^2 - \left(C^p_{2211}\Gamma_{12} + C^p_{1122}\Gamma_{21}\right) n_1 n_2 + C^p_{2222}\Gamma_{11} n_2^2 = 0 \\ a_1 &= \det\left(\Gamma_{ik}\right) \end{aligned} \tag{99}$$

In view of Eqs. (98) and (99) we make the following remarks:

1. The condition

$$a_1 = \det(\Gamma_{ik}) = 0 \tag{100}$$

coincides with the classical bifurcation condition (Vardoulakis & Sulem, 1995) and, if evaluated at the first occurring bifurcation state C_B, it yields the shear-band orientation.

1) The quadratic form $a_1\left(n_1, n_2\right)$ changes sign at the bifurcation state of the classical continuum,

$$a_1(n_1, n_2) = \begin{cases} \geq 0 \text{ for} : \Psi \leq \Psi_B \\ < 0 \text{ for} : \Psi > \Psi_B \end{cases} \tag{101}$$

2) The quadratic form $a_0\left(n_1, n_2\right)$ determines the type of the governing partial differential equations. In order to regularize the original problem we require that

$$a_0(n_1, n_2) > 0 \tag{102}$$

With this condition the system of partial differential equations (76) is always elliptic, as opposed to the original system of governing equations ($\ell = 0$), which is of changing type, namely turning from elliptic to hyperbolic at the point of shear-band bifurcation. Following these observations we conclude that:

1. Prior to shear-band bifurcation there is no real solution for the shear-band thickness; i.e. there exist no localization solution.
1) At the bifurcation state the shear-band thickness is infinite as compared to the material length $\ell > 0$, and is rapidly decreasing in the post-bifurcation regime (Fig. 4),

$$(Q\ell) = \sqrt{-\frac{a_1(n_1, n_2)}{a_0(n_1, n_2)}} \tag{103}$$

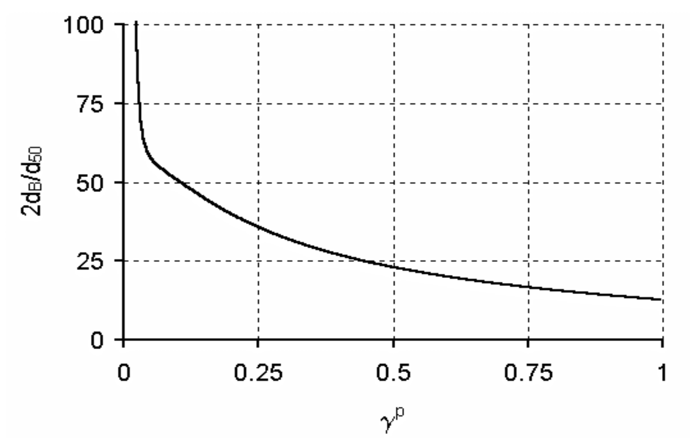

Figure 4. Example of estimated shear-band thickness as function of softening rate in a sand sp ecimen tested in plane-strain compression ($\ell_c = d_{50} = 0.33$ mm) (Vardoulakis & Aifantis, 1989)

The above illustrated shear-band analysis is used for estimating the internal length parameter ℓ_c . Empirical evidence is suggesting that shear-band thickness correlates to the mean grain size, as shown below in Fig. 5.

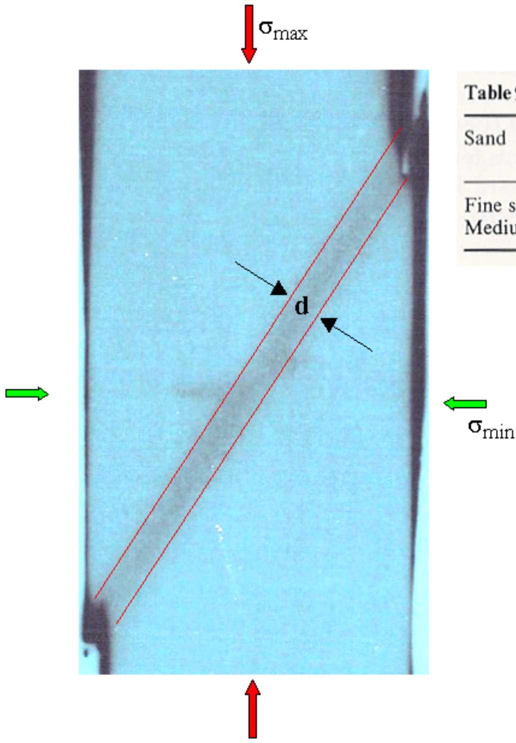

Table 9.4.1 Measured shear-band thickness (Vardoulakis *et al.*, 1985a,b)

Sand	$d_{50\%}$ (mm)	$2d_E$ (mm)	$2d_E/d_{50\%}$ (—)
Fine sand (FS)	0.20	3.7	18.5
Medium sand (D)	0.33	4.3	13.0

$$\boxed{d \approx 10 \div 20\ D_{50}}$$

Figure 5. Shear-band thickness correlation to grain size (Vardoulakis, 1988).

4.6 Boundary Localizations

When a granular material is sheared against a *rough* boundary, zones of localized deformation are observed at the interface. This happens for example along the walls of silos and at the interface between soil and pile driven into it. Fig. 6 below shows an x-ray plate of a shear-interface test performed in the biaxial apparatus (Tejhman & Wei Wu, 1995).

Following the paper by Bogdanova-Bontcheva & Lippmann (1975) and Unterreiner & Vardoulakis (1994), controlled interface shear tests on granular materials were also performed recently by Lerat et al. (1997), in a newly developed Ring-Shear Apparatus, Figs. 7 and 8. The boundary localization phenomenon in the Ring-Shear Apparatus was simulated by Zervos et al. (2000) by using a contact dynamics program, Fig. 9. These numerical tests have shown that, for a 2-D Schneebeli material, at the interface localization the rolling contacts are at least twice as frequent as sliding contacts (Fig. 10).

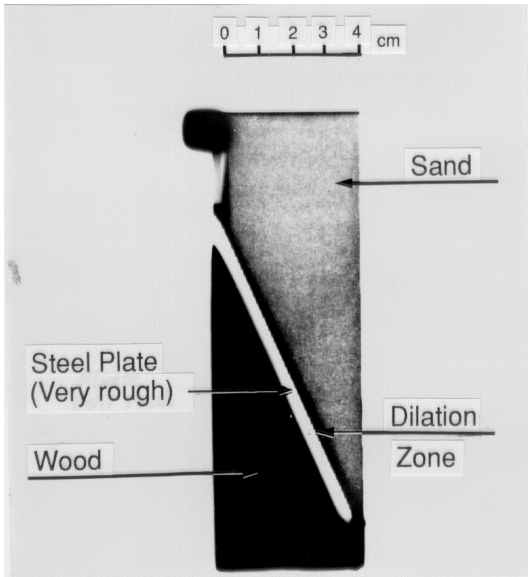

Figure.6. Interface softening dilatancy localization for a rough steel plate interfacing with sand in the biaxial apparatus (Tejhman & Wei Wu,1995).

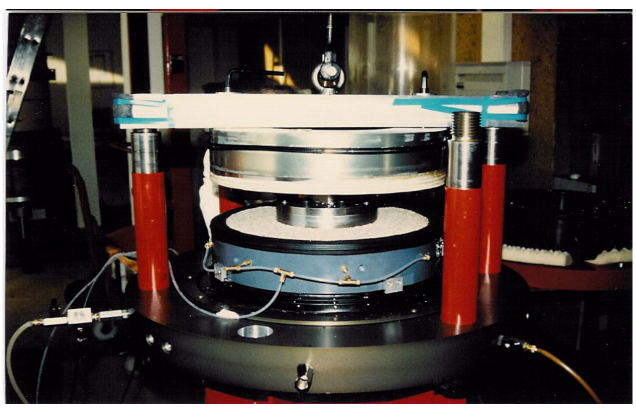

Figure 7. Plane-strain Couette apparatus for sand (Corfdir, A., Lerat, P. and Vardoulakis, I. (2003). A cylinder shear apparatus. Geotechnical Testing Journal, submitted.)

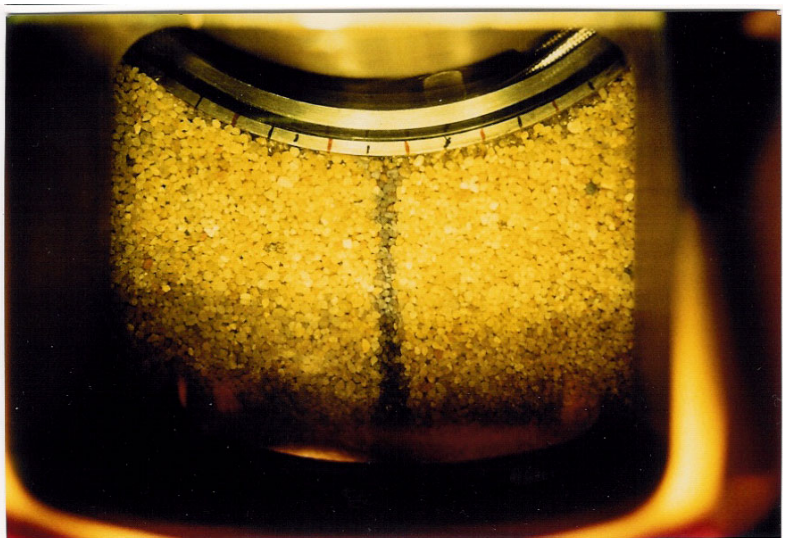

Figure 8. Interface localization in granular material realized in the Ring Shear Apparatus [2]

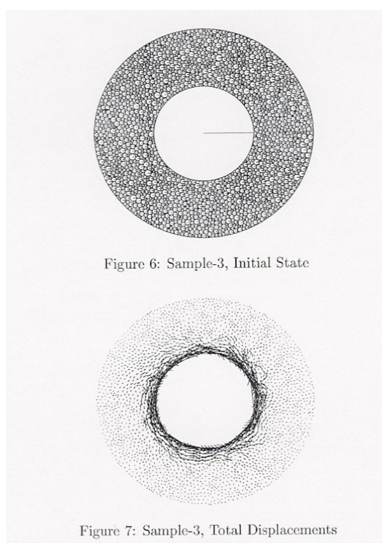

Figure 9. Contact dynamics simulation of interfacial localization (Zervos et al. 2000)

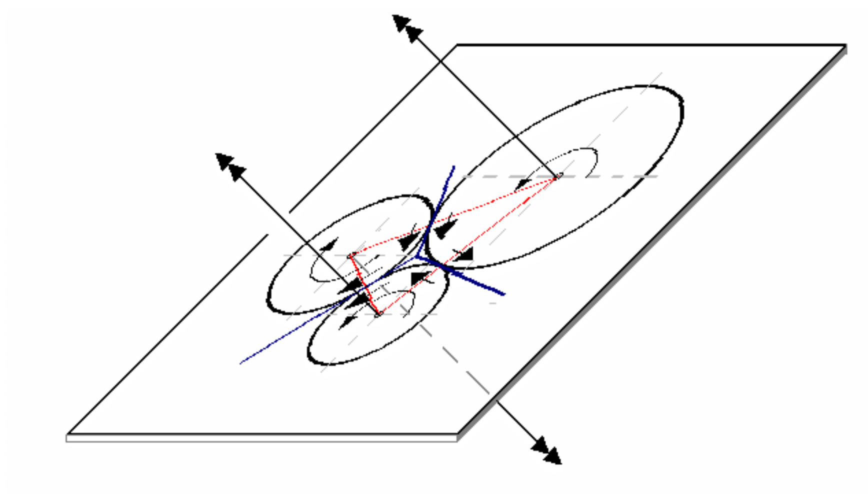

Figure 10. In this configuration we see that for each sliding contact two rolling contacts are necessary, an effect usually called coarse graining. This figure illustrates the necessity of extending the classical Cosserat granular model of Mühlhaus & Vardoulakis (1987) so as to include bipolar effects as well.

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Appendix I: The Schrödinger operator

We consider Eq. (54)

$$(1-\ell^2\nabla^2)\dot{\Psi} = \Lambda \quad , \quad \Lambda = \frac{1}{H}B_{kl}D_{kl} \tag{A.1}$$

Notice that in the literature the operator

$$S \approx 1-\ell^2\nabla^2 \tag{A.2}$$

is known in the literature as the Schrödinger operator.

We assume for simplicity the 1D-case

$$\nabla^2 \approx \frac{\partial^2}{\partial y^2} \tag{A.3}$$

where y is for example the direction along which the fields vary rapidly; i.e. the direction normal to the localization direction (Fig. 2).

We consider the Fourier transforms of the functions $\dot{\Psi}(y)$ and $\Lambda(y)$ as well as of their derivatives up to order two exist. In that case the functions $\dot{\Psi}$, $\dot{\Psi}'$ and $\dot{\Psi}''$ are continuous and $\dot{\Psi} \to 0$ as $|y| \to \infty$; the same is assumed to hold also for the function Λ. Thus we set,

$$Y(\alpha) = E\{\dot{\Psi}(y)\} = \int_{-\infty}^{+\infty} \dot{\Psi}(y) e^{-i\alpha y} dy \quad , \quad L(\alpha) = E\{\Lambda\} \int_{-\infty}^{+\infty} \Lambda(x)\, e^{-i\alpha y} dy \tag{A.4.1}$$

$$\dot{\Psi}(y) = E^{-1}\{Y(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(\alpha) e^{i\alpha y} d\alpha \quad , \quad \Lambda(y) = E^{-1}\{L(\alpha)\} \int_{-\infty}^{+\infty} L(\alpha) e^{i\alpha y}\, d\alpha \tag{A.4.2}$$

If we apply the Fourier Transformation on both sides of Eq. (A.1) we get,

$$Y - \ell^2 \alpha^2 Y = L \quad \Rightarrow \quad Y = \frac{L}{1 - (\alpha\ell)^2} = \left(1 + (\alpha\ell)^2\right) L + O\left((\alpha\ell)^4\right) \tag{A.5}$$

By applying the inverse Fourier transform on the last equation we get,

$$E^{-1}\{Y\} \approx E^{-1}\{L\} + \ell^2 E^{-1}\{\alpha^2 L\}$$

or

$$\dot{\Psi} \approx \Lambda + \ell^2 \frac{d^2\Lambda}{dy^2} \tag{A.6}$$

Thus for slow varying constitutive properties from Eq. (A.6) we get Eq. (54).