Chapter 3:
Linear Micro-elasticity

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Abstract. This chapter provides a brief introduction to the theory of linear Elasticity with micro-structure.

3.1 Introduction

Fifty years after the first publication of the original work of the brothers Cosserat, Eugéne and François in 1909, the basic kinematic and static concepts of the "Cosserat" continuum were reworked in a milestone paper by late Professor Günther (1958). Günther's paper marks the rebirth of continuum micro-mechanics in the late 50's and early 60's. Following this publication, several hundred papers were published all over the world on that subject. A variety of names have been invented and given to theories of various degrees of rigor and complexity: They were called "Cosserat continua" or "micro-polar media", "oriented media", "continuum theories with directors", "multi-polar continua", "micro-structured" or "micro-morphic continua", etc (cf. Herrmann, 1972). The state-of-the-art at this time was reflected in the collection of papers presented at the historical IUTAM Symposium on the *"Mechanics of Generalized Continua"*, in Freudenstadt and Stuttgart in 1967.

On the subject of Cosserat Elasticity recommendable for their clarity and didactical value are the papers by H. Schaeffer (1962, 1967) and by Kessel (1964) in German and the paper of Koiter (1964) in English. Notable and of equal importance in relation to Gradient Elasticity is the milestone paper by Mindlin (1964) and two papers by Germain (1973a &b), the latter written partially in French and partially in English.

In this chapter we present a simple version of Mindlin's linear Elasticity theory with microstructure. Exadaktylos (1998) Exadaktylos et al. (1996, 1998, 2001 a & b), Georgiadis et al. (1998, 2000, 2001, 2002), and Vardoulakis & Sulem (1995) have applied this theory in a number papers, where static and dynamic boundary-value problems have been addressed and solved analytically[1] and numerically by Amanatidou & Aravas (2002). Here the static micro-elasticity theory is critically reconsidered. The basic reference for the micro-elasticity theory given herein, is the paper by Mindlin & Eshel (M&E, 1968). Central to our discussion is the distinction between the 'balance' stresses, which appear in the formulation of force- and moment-equilibrium equations, and the 'constitutive' stresses, which are introduced through the variation of the elastic strain energy density function. The existence of higher order stresses such as couple- and double-stresses is demonstrated and the relations between equilibrium and

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[1] See also additional references to the current research literature at the end of this chapter.

constitutive stresses are derived. The uniqueness theorem proof presented here follows the line of the proof given by M&E and is a variant of the Kirchhoff uniqueness argument. The uniqueness theorem proof is based on the hypothesis for positive elastic strain energy density function, which in our case is guarantied as soon as some constitutive inequalities are satisfied; cf. Georgiadis et al. (2002). We close our discussion here by considering the issue of boundary conditions, which is central to any higher order continuum theory and we try to clarify the issue by discussing some simple boundary-value problems.

3.2 The Constitutive Equations of the "Casal-Mindlin" Micro-elasticity: The strain energy density function

As a starting point for our presentation we select here a rather unnoticed early publication by Casal (1961), which is referenced in the aforementioned papers by Germain. Casal's model is one-dimensional and starts from postulating the total elastic strain-energy in a tension bar with stiffness (EA) and length L

$$W = \frac{1}{2}\int_0^L (EA)\left(\varepsilon^2 + \ell(\nabla\varepsilon)^2\right)dx + \frac{1}{2}\left((EA)\ell'\varepsilon^2\right)\Big|_0^L \tag{1}$$

In Eq. (1) one can easily recognize the classical term, pertaining to the contribution of strain, $\varepsilon = \nabla u$. On top of that, Eq. (1) introduces a strain-gradient effect in the volume-part as well as on the boundaries of the bar. Casal's tension bar is characterized by the introduction of two additional material constants on top of the Young's modulus E. These additional constants, ℓ and ℓ', have the dimension of length, and are responsible for the corresponding volume and surface strain-gradient dependent elastic energy contributions.
Motivated by Casal's original idea, we consider here and in our previous work the following special form for the elastic strain-energy density function (Vardoulakis & Sulem 1995)

$$\begin{aligned}\hat{w} = {} & \frac{1}{2}\lambda\varepsilon_{mm}\varepsilon_{nn} + G\varepsilon_{mn}\varepsilon_{nm} + \ell^2\left(\frac{1}{2}\lambda\hat{\kappa}_{kmm}\hat{\kappa}_{knn} + G\hat{\kappa}_{kmn}\hat{\kappa}_{knm}\right) + \\ & \ell'_k\left(\frac{1}{2}\lambda\left(\hat{\kappa}_{kmm}\varepsilon_{nn} + \varepsilon_{mm}\hat{\kappa}_{knn}\right) + G\left(\hat{\kappa}_{kmn}\varepsilon_{nm} + \varepsilon_{mn}\hat{\kappa}_{knm}\right)\right)\end{aligned} \tag{2}$$

In Eq. (2), ε_{ij} and $\hat{\kappa}_{ijk}$ denote the infinitesimal strain and strain gradient, respectively

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \ , \quad \hat{\kappa}_{ijk} = \hat{\kappa}_{ikj} = \partial_i \varepsilon_{jk} \tag{3}$$

where $u_i = u_i(x_k, t)$ is the (infinitesimal) displacement field.

As compared to linear isotropic elasticity the present micro-elasticity theory is equipped by two additional constitutive parameters: the internal length ℓ and the director length ℓ'_k. If we set these to length parameters equal to zero we regain Hooke's law. Thus in Eq. (2) λ and G are identified as the Lamé constants of the material. As already mentioned this theory is a special case of a Mindlin-type 2nd gradient linear Elasticity theory with micro-structure[2]. According to the classification given by M&E the considered form of the elastic energy density function, Eq. (2), is falling into the category of energy functions of the 2nd form, which depend only on the strain and its gradient.

$$\hat{w} = \hat{w}(\varepsilon_{ij}, \hat{\kappa}_{ijk}) \tag{4}$$

We recall the 1st form of elastic strain energy density function after M&E

$$\tilde{w} = \tilde{w}(\varepsilon_{ij}, \tilde{\kappa}_{ijk}) \quad , \quad \tilde{\kappa}_{ijk} = \partial_i \partial_j u_k \tag{5}$$

and the 3rd form

$$\overline{w} = \overline{w}(\varepsilon_{ij}, \overline{\kappa}_{ij}, \overline{\overline{\kappa}}_{ijk}) \;,\; \overline{\kappa}_{ij} = \frac{1}{2}\varepsilon_{jlk}\partial_l \partial_i u_k \quad , \quad \overline{\overline{\kappa}}_{ijk} = \frac{1}{3}\left(\partial_i \partial_j u_k + \partial_j \partial_k u_i + \partial_k \partial_i u_j\right) \tag{6}$$

As is shown in M&E all three forms for the strain energy density function are equivalent. Using the first variation of the elastic energy density function we can introduce stress-like quantities which are dual in energy to the strain and to the strain-gradient respectively,

$$\hat{w} = \hat{w}(\varepsilon_{ij}, \hat{\kappa}_{ijk}) \quad \Rightarrow \quad \delta\hat{w} = \hat{\tau}_{ij}\delta\varepsilon_{ij} + \hat{\mu}_{ijk}\delta\hat{\kappa}_{ijk} \tag{7}$$

The stresses $\hat{\tau}_{ij}$ and $\hat{\mu}_{ijk}$ are called here **constitutive stresses** and are not to be confused with the balance stresses, which are introduced through the momentum and angular momentum balance laws. As we will see in this section, unlike in classical Elasticity Theory the above-defined constitutive stresses, Eqs. (7), do not coincide with the balance stresses.

From Eqs. (2) and (7) we get

$$\hat{\tau}_{ij} = \frac{\partial \hat{w}}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} + \ell'_k\left(\lambda\hat{\kappa}_{knn}\delta_{ij} + 2G\hat{\kappa}_{kij}\right) \tag{8}$$

$$\hat{\mu}_{ijk} = \frac{\partial \hat{w}}{\partial \hat{\kappa}_{ijk}} = \ell^2\left(\lambda\hat{\kappa}_{inn}\delta_{jk} + 2G\hat{\kappa}_{ijk}\right) + \ell'_i\left(\lambda\varepsilon_{nn}\delta_{jk} + 2G\varepsilon_{jk}\right) \tag{9}$$

[2] Notice that Mindlin's general elastic strain energy function has 903 independent coefficients!

3.3 Balance Stresses

As already mentioned balance stresses are defined independently of any constitutive equations, through the formulation of the balance laws of linear and angular momentum. In order to formulate these balance laws we must postulate the external forces which act on the considered continuum. Accordingly for the considered micro-elastic continuum we assume the existence of the following set of external forces:

- body forces, $F_k dV$
- surface tractions, $t_k dS$
- surface couples with moment $m_k dS$
- self-equilibrating surface double tractions, $R_{ik} dS$

The set forces $\{F_k, t_k\}$ appear in the balance of linear and angular momentum whereas the couples m_k appear only in the balance of angular momentum. The self-equilibrating double tractions R_{ik} do not appear in the balance laws. The body forces $F_i(x_k)$ could be for example gravity-field forces, acting per unit volume of the material; e.g. $F_i = \rho g_i$, where $\rho(x_k, t)$ is the mass density field and g_i the acceleration of gravity. Surface tractions are related to a non-symmetric "balance" stress tensor field $\tau_{ij}(x_k, t)$, and to the unit outward normal vector n_i on the considered surface element on which they act

$$t_i = n_k \tau_{ki} \tag{10}$$

In the static case balance of linear momentum is expressed by the following global force equilibrium condition,

$$\int_V F_i dV + \int_{\partial V} t_i dS = 0 \tag{11}$$

This condition together with Eq. (10) and Gauss' theorem yields the corresponding local equilibrium conditions for the balance stress tensor

$$\int_V (\partial_j \tau_{jk} + F_k) dV = 0 \quad \Rightarrow \quad \underline{\partial_j \tau_{jk} + F_k = 0} \tag{12}$$

Thus in the static limit, the principle of conservation of linear momentum reduces to the equilibrium equations for the non-symmetric balance stress tensor (Fig. 1). In other words the above equilibrium Eq. (12) gives the defining property of the tensor field $\tau_{ij}(x_k, t)$, as that 2nd order tensor field, which is in force equilibrium with the body force field.

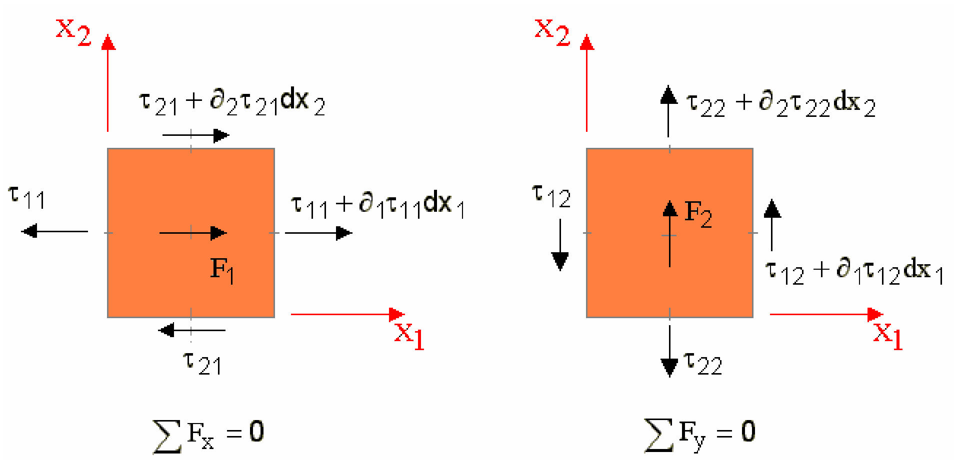


Figure 1. Force equilibrium on an infinitesimal material element

Similarly we define a couple-stress tensor field $\mu_{ij}(x_k,t)$, which is relates the surface couples m_i to the corresponding outward unit vector on the surface element on which they act

$$m_i = n_k \mu_{ki} \tag{13}$$

In the static limit balance of angular momentum is expressed by the following global moment equilibrium condition,

$$\int_V \varepsilon_{ijk} x_j F_k dV + \int_{\partial V} (\varepsilon_{ijk} x_j t_k + m_i) dS = 0 \tag{14}$$

where ε_{ijk} is the Levi-Civita permutation tensor. Eq. (14), together with Eq. (13) and Gauss' theorem yield the local moment equilibrium conditions for the balance couple-stress tensor (Fig. 2):

$$\int_V \varepsilon_{ijk} x_j \partial_l \tau_{lk} dV + \int_{\partial V} (\varepsilon_{ijk} x_j n_l \tau_{lk} + n_k \mu_{ki}) dS = 0$$

or

$$\int_V (\varepsilon_{ijk} x_j \partial_l \tau_{lk} - \partial_l (\varepsilon_{ijk} x_j \tau_{lk}) - \partial_k \mu_{ki}) dV = 0 \quad \Rightarrow$$

$$\partial_j \mu_{jk} + \varepsilon_{ijk} \tau_{ij} = 0 \tag{15}$$

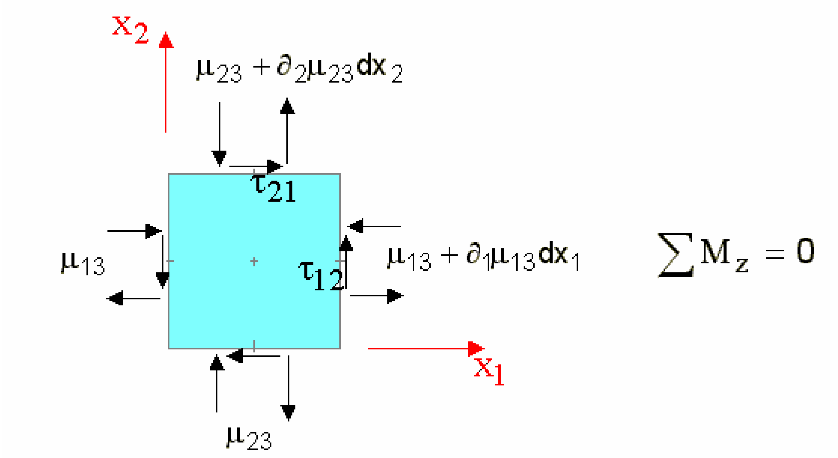


Figure 2. Moment equilibrium on an infinitesimal material element

Thus equilibrium Eq. (15) gives the defining property of the tensor field $\mu_{ij}(x_k, t)$, as that 2nd order tensor field, which is in moment equilibrium with the antisymmetric part of the balance stress tensor.

In order to formulate the energy balance equation, we assume that the body forces and the surface tractions are working on the velocity field $v_i \approx \partial_t u_i$ and that the surface couples are working rate of the rotation vector,

$$\dot{w}_i = \frac{1}{2} \varepsilon_{ijk} \partial_j v_k \tag{16}$$

Finally we assume that the self-equilibrating surface double-tractions R_{ij} are defined by the 3rd order tensor field of the “balanced” double-stresses $\mu_{ijk}(x_l, t)$,

$$R_{ij} = n_k \mu_{kij} \tag{17}$$

These double tractions are assumed to be working on the corresponding (symmetric) strain-rate tensor

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \tag{18}$$

According to the above definitions the power of the external forces is defined by the following expression

$$W^{ext} = \int_V F_k v_k dV + \int_{\partial V} \left(t_k v_k + m_k \dot{w}_k + R_{ij}\dot{\varepsilon}_{ij}\right) dS \tag{19}$$

We decompose additively the true stress in a symmetric part $\tau_{(ij)}$ and in an anti-symmetric part $\tau_{[ij]}$

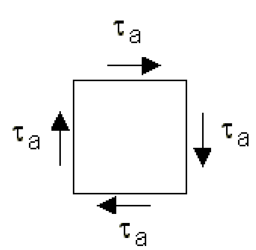
$$\tau_{ij} = \tau_{(ij)} + \tau_{[ij]}$$

$$\tau_{(ij)} = \frac{1}{2}(\tau_{ij} + \tau_{ji}) \quad , \quad \tau_{(ij)} = \tau_{(ji)} \tag{20}$$

$$\tau_{[ij]} = \frac{1}{2}(\tau_{ij} - \tau_{ji}) \quad , \quad \tau_{[ij]} = -\tau_{[ji]}$$

The moment equilibrium Eq. (15) becomes (Fig. 3),

$$\tau_{[ik]} = -\tfrac{1}{2}\varepsilon_{ikm}\partial_l \mu_{lm} \tag{21}$$



$$\tau_{[21]} = \tau_a = \frac{1}{2}(\partial_1 \mu_{13} + \partial_2 \mu_{23})$$

$$\tau_{[12]} = -\tau_a$$

Figure 3. The antisymmetric part of the balance stress tensor

Further we split the couple stress tensor into its spherical and deviatoric part

$$\mu_{ij} = \mu_{ij}^{D} + \frac{1}{3}\mu_{kk}\delta_{ij} \tag{22}$$

and we introduce this in Eq. (21)

$$\tau_{[ik]} = -\tfrac{1}{2}\varepsilon_{ikm}\partial_{l}\mu_{lm}^{D} \tag{23}$$

Then by using the stress decomposition Eq. (19.1) we get

$$\tau_{ij} = \tau_{(ij)} - \tfrac{1}{2}\varepsilon_{ikm}\partial_{l}\mu_{lm}^{D} \tag{24}$$

and with that the force equilibrium Eq. (12) becomes

$$\underline{\partial_{i}\tau_{(ik)} + F_{k} = \frac{1}{2}\varepsilon_{ikm}\partial_{i}\partial_{l}\mu_{lm}^{D}} \tag{25}$$

For elastic materials and for isothermal processes, the principle of energy conservation postulates that the power of external forces must be in equilibrium with the rate of the total elastic-strain energy

$$\underline{\dot{W}^{el} = W^{ext}} \quad \Rightarrow \quad \int_{V}\dot{\hat{w}}dV = \int_{V}F_{k}v_{k}dV + \int_{\partial V}\left(t_{k}v_{k} + m_{k}\dot{w}_{k} + R_{ij}\dot{\varepsilon}_{ij}\right)dS \tag{26}$$

with (cf. Eq. 7)

$$\dot{\hat{w}} = \hat{\tau}_{ij}\dot{\varepsilon}_{ij} + \hat{\mu}_{ijk}\delta\dot{\hat{\kappa}}_{ijk} \tag{27}$$

By using the divergence theorem and the definitions of kinematic variables from Eqs. (26) and (27) we obtain the following relations[3]:

$$\hat{\tau}_{jk} = \tau_{(jk)} + \partial_{i}\mu_{ijk} \tag{28}$$

$$\hat{\mu}_{ijk} = \mu_{ijk} + \frac{1}{2}\left(\varepsilon_{lik}\mu_{jl} + \varepsilon_{lij}\mu_{kl}\right) \tag{29}$$

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[3] The proof of Eq. (31) can be found in the paper by M&E. For the proof of Eqs. (32) to (34) see also in Appendix I. Appendices I, II and III of this section are based on the paper by M&E and have been worked out by Mrs Lilian Vougiouka, as a part of her Master of Civil Engineering Thesis at NTU Athens in November 2001.

These relations are connecting the constitutive stresses $\hat{\tau}_{jk}$, $\hat{\mu}_{ijk}$, which are dual in energy to the strain and its gradient, to the balance stresses τ_{jk}, μ_{ijk}, which are defined by the equations of force and moment equilibrium. We eliminate μ_{ijk} from Eqs. (28) and (29) and we get

$$\tau_{(jk)} = \hat{\tau}_{jk} - \partial_i \hat{\mu}_{ijk} + \frac{1}{2}\partial_i \mu^D_{jl} \varepsilon_{lik} + \frac{1}{2}\partial_i \mu^D_{kl} \varepsilon_{lij} \tag{30}$$

We can prove the following relationships[4]

$$\mu^D_{ij} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{jpq} \tag{31}$$

$$\overline{\mu}_{ij} = \mu^D_{ij} \tag{32}$$

$$\mu_{ijk} = \frac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}) \tag{33}$$

$$\overline{\mu}_{il}\varepsilon_{ljk} + \overline{\mu}_{jl}\varepsilon_{lki} + \overline{\mu}_{kl}\varepsilon_{lij} = 0 \tag{34}$$

We define now the following symmetric tensor

$$\underline{\sigma_{jk} = \hat{\tau}_{jk} - \partial_i \hat{\mu}_{ijk}} \tag{35}$$

Substituting Eqs. (30) to (35) into Eq. (25) we get that the equilibrium equations can be written in terms of the above introduced tensor σ_{ij},

$$\underline{\partial_j \sigma_{jk} + F_k = 0} \tag{36}$$

For this reason the stress-tensor σ_{ij} will be called here the equilibrium stress tensor**.**

By using the constitutive Eqs. (8) and (9) for the constitutive stresses $\hat{\tau}_{ij}$ and $\hat{\mu}_{ijk}$ and the definition of the balance stress σ_{ij}, Eq. (34), we get that

$$\sigma_{jk} = \left(1 - \ell^2 \nabla^2\right)\left(\lambda \varepsilon_{ii} \delta_{jk} + 2G\varepsilon_{jk}\right) \tag{37}$$

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[4] Recall that $\overline{\mu}_{ij} = \dfrac{\partial \overline{w}}{\partial \overline{\kappa}_{ij}}$, where $\overline{w} = \overline{w}(\varepsilon_{ij}, \overline{\kappa}_{ij}, \overline{\overline{\kappa}}_{ijk})$ is M&E's 3rd form of the strain energy density function (Appendix II).

With this expression for the balance stress the equilibrium condition (36) yields,

$$\partial_j\left(1-\ell^2\nabla^2\right)\left(\lambda\varepsilon_{ii}\delta_{jk}+2G\varepsilon_{jk}\right)+F_k=0 \tag{38}$$

or

$$\lambda\delta_{jk}\partial_j\partial_i\left(u_i-\ell^2\partial_l\partial_l u_i\right)+G\left(\partial_j\partial_j\left(u_k-\ell^2\partial_l\partial_l u_k\right)+\partial_j\partial_k\left(u_j-\ell^2\partial_l\partial_l u_j\right)\right)+F_k=0 \tag{39}$$

Exadaktylos & Vardoulakis (2001) showed that the general solution of Eq. (39) may be expressed in terms of generalized Neuber-Papkovich displacement functions.

Remarks

We may re-interpret Eq. (37) as follows: We assume the existence of a local equilibrium stress σ_{ij}

$$\partial_j\sigma_{jk}+F_k=0$$

We assume that Hooke's law does not hold for the local equilibrium stress but an average stress, defined over a representative elementary volume:

$$<\sigma_{ij}>=\int_{(REV)}\sigma_{ij}dV=\lambda\varepsilon_{kk}\delta_{ij}+2G\varepsilon_{ij}$$

We can easily show that for cubic (REV) with dimension a the following approximation formula holds:

$$<\sigma_{ij}>_{(REV)}=\left(1+\frac{a}{24}^2\nabla^2\right)\sigma_{ij}(x_k)$$

By inverting the differential operator we get Eq. (37):

$$\sigma_{ij}=\frac{1}{1+\frac{a^2}{24}\nabla^2}<\sigma_{ij}>_{(REV)}\approx\left(1-\frac{a^2}{24}\nabla^2\right)\left(\lambda\varepsilon_{kk}\delta_{ij}+2G\varepsilon_{ij}\right)$$

We remark that the governing differential Eq. (38), that guaranties equilibrium, is written in terms of the strain tensor ε_{ij} and involves only the material length ℓ. Thus for the determination of the displacement field $u_k(x_i)$ we do not need to specify the spatial distribution of the director field $\ell'_i(x_k)$, which is appearing in the strain-energy density function, Eq. (2). As we will see below, the director field ℓ'_k needs only to be specified on the boundary of the considered micro-elastic body. This property of the considered microelasticity model is indeed reminiscent of the problem of surface tension T in fluids. For example in the study of capillary surface waves in fluids the information concerning the characteristic length of the problem, $\ell' = 2\pi\sqrt{T/(\rho g)}$, enters only through the dynamic boundary condition, that assigns on the surface of the fluid a fictitious pressure that is proportional to the mean curvature of that surface[5], $\Delta p = -T\,\nabla^2\zeta(x,y)$.

3.4 Relations Between Balance- and Constitutive Stresses

We start from the decomposition of the true tensor in symmetric and antisymmetric part, Eq. (20). From the derivations presented in the previous section we get that,

$$\tau_{(jk)} = \sigma_{jk} + \frac{1}{2}\partial_i\mu^D_{jl}\varepsilon_{lik} + \frac{1}{2}\partial_i\mu^D_{kl}\varepsilon_{lij} \tag{40}$$

$$\tau_{[ik]} = -\frac{1}{2}\varepsilon_{ikm}\partial_l\mu_{lm} \tag{41}$$

$$\mu_{ik} = \mu^D_{ik}, \quad \underline{\mu_{nn} = 0} \tag{42}$$

The assumption that the spherical part of the couple-stress tensor is vanishing ($\mu_{nn} = 0$) is justified by the fact that it can be proven[6] that the solution of the generic boundary-value problem is independent of the value of μ_{nn}. Moreover it holds,

$$\mu^D_{ij} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{jpq} \tag{43}$$

with $\hat{\mu}_{ijk}$ given by Eq. (9) in terms of the strain and its gradient. Thus we get the following relationship between the balance stress and the constitutive stresses,

$$\tau_{jk} = \hat{\tau}_{jk} - \frac{1}{3}\partial_i(\hat{\mu}_{ijk} + 2\hat{\mu}_{jki}) \tag{44}$$

or explicitly in terms of the strain and the strain gradient,

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[5] cf. L. A. Segel, *Mathematics Applied to Continuum Mechanics,* Dover, 1977.

[6] see Sect. 3.5

$$\tau_{jk} = (1-\frac{1}{3}\partial_i\ell'_i)(\lambda\varepsilon_{nn}\delta_{jk} + 2G\varepsilon_{jk}) - \frac{2}{3}\partial_i\ell'_j(\lambda\varepsilon_{nn}\delta_{ki} + 2G\varepsilon_{ki}) + \\ + \frac{1}{3}\Big\{\lambda\Big[(2\ell'_i - \ell^2\partial_i)\delta_{jk} - 2(\ell'_j + \ell^2\partial_j)\delta_{ki}\Big]\hat{\kappa}_{inn} + \\ + 2G\Big[(2\ell'_i - \ell^2\partial_i)\hat{\kappa}_{ijk} - 2(\ell'_j + \ell^2\partial_j)\hat{\kappa}_{iki}\Big]\Big\} \tag{45}$$

Similarly for the balance couple stress tensor we get the following expressions: a) in terms of constitutive stresses:

$$\mu_{ik} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{kpq} \tag{43}$$

or b) in terms of strains and strain gradients

$$\mu_{ik} = \frac{2}{3}\Big\{\lambda(\ell^2\hat{\kappa}_{pnn} + \ell'_p\varepsilon_{nn})\delta_{iq} + 2G(\hat{\kappa}_{ipq} + \ell'_i\varepsilon_{pq} + \hat{\kappa}_{piq} + \ell'_p\varepsilon_{iq})\Big\}\varepsilon_{kpq} = \\ = \frac{2}{3}\Big\{\lambda(\ell^2\partial_p + \ell'_p)\varepsilon_{nn}\delta_{iq} + 2G\Big[(\partial_i + \ell'_i)\varepsilon_{pq} + (\partial_p + \ell'_p)\varepsilon_{iq}\Big]\Big\}\varepsilon_{kpq} \tag{44}$$

The balanced double stresses may be expressed again in terms of constitutive stresses and strains, as follows[7]:

$$\mu_{ijk} = \frac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}) \tag{45}$$

or

$$\mu_{ijk} = \frac{1}{3}\Big\{\lambda\Big[(\ell^2\partial_i + \ell'_i)\delta_{jk} + (\ell^2\partial_j + \ell'_j)\delta_{ki} + (\ell^2\partial_k + \ell'_k)\delta_{ij}\Big]\varepsilon_{nn} + \\ 2G\Big[(\partial_i + \ell'_i)\varepsilon_{jk} + (\partial_j + \ell'_j)\varepsilon_{ki} + (\partial_k + \ell'_k)\varepsilon_{ij}\Big]\Big\} \tag{46}$$

3.5 A Uniqueness Theorem

In this section the uniqueness of the solution of the general boundary value problem as stated in the previous sections is discussed. The equations for the static boundary value problem of the 2nd gradient elasticity, based on M&E's 2nd form of the elastic strain energy-density function $(\overline{w} = \hat{w} = \overline{\overline{w}})$ are:

- *Equilibrium equations (balance of linear and angular momentum)*

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[7] Appendix II

$\{3\}$: $\partial_j \tau_{jk} + F_k = 0$ *(true stress equilibrium)* (47.1)

$\{3\}$: $\partial_j \mu_{jk} + \varepsilon_{ijk}\tau_{ij} = 0$ or $\tau_{[ij]} = -\frac{1}{2}\varepsilon_{ijk}\partial_l \mu_{lk}$ *(true couple stress equilibrium)* (47.2)

- *Compatibility equations*

$\{6\}$: $\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$ *(strain)* (47.3)

$\{3\}$: $w_i = \frac{1}{2}\varepsilon_{ijk}\partial_j u_k$ *(rotation)* (47.4)

$\{18\}$: $\hat{\kappa}_{ijk} = \hat{\kappa}_{ikj} = \partial_i \varepsilon_{jk} = \frac{1}{2}\left(\partial_i \partial_j u_k + \partial_i \partial_k u_j\right)$ *(strain gradient)* (47.5)

- *Constitutive equations*

$\{6\}$: $\hat{\tau}_{ij} = \dfrac{\partial \hat{w}}{\partial \varepsilon_{ij}} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} + \ell'_k\left(\lambda \kappa_{knn}\delta_{ij} + 2G\kappa_{kij}\right)$ (47.6)

$\{18\}$: $\hat{\mu}_{ijk} = \dfrac{\partial \hat{w}}{\partial \hat{\kappa}_{ijk}} = \ell^2\left(\lambda \hat{\kappa}_{inn}\delta_{jk} + 2G\kappa_{ijk}\right) + \ell'_i\left(\lambda \varepsilon_{nn}\delta_{jk} + 2G\varepsilon_{jk}\right)$ (47.7)

- *Restrictions due to energy balance*

$\{6\}$: $\tau_{(jk)} = \hat{\tau}_{jk} - \partial_i \mu_{ijk} = \hat{\tau}_{jk} - \partial_i \hat{\mu}_{ijk} + \frac{1}{2}\varepsilon_{lik}\partial_i \mu^D_{jl} + \frac{1}{2}\varepsilon_{lij}\partial_i \mu^D_{kl}$ (47.8)

$\{8\}$: $\mu^D_{ij} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{jpq}$ or $\mu^D_{ij} = \frac{2}{3}\hat{\mu}_{kpi}\varepsilon_{jkp}$ (47.9)

These are 71 equations for 72 dependent field variables, namely:

- (3) u_i, (3) w_i, (6) ε_{ij}, (18) $\hat{\kappa}_{ijk}$, (9) τ_{ij}, (9) μ_{ij}, (6) $\hat{\tau}_{ij}$, (18) $\hat{\mu}_{ijk}$.

The vector of unknown dependent variables is denoted by,

$$X = \{u_i, w_i, \hat{\kappa}_{ijk}, \tau_{ij}, \mu_{ij}, \hat{\tau}_{ij}, \hat{\mu}_{ijk}\}^T \tag{48}$$

The additional variable is the spherical part of the couple stress, which we call the mean torsion,

$$\mu = \frac{1}{3}\mu_{kk} \tag{49}$$

The mean torsion does not contribute to the elastic stain energy-density and is indeterminate within the framework of the considered micro-elasticity theories, represented by the above 71 equtions (47.1-9).

For the proof of uniqueness of the solution of the general boundary-value problem, we will consider that there are two different solution vectors X' and X'' with μ' and μ'' arbitrary.

Let their difference be $X = X' - X''$. If each set is a solution so is their difference. For this set X from Eq. (47.1) follows

$$\int_{t_0}^{t} dt \int_V \Pi \, dV = 0 \quad , \quad \Pi = \left(\partial_i \tau_{ij} + F_j\right)\dot{u}_j \tag{50}$$

where at time t_0 the body is unstrained and at time t the total loading has been accomplished. In Eq. (50) the triple integral is over the volume V with boundary surface ∂V of the considered body **B**. With $\dot{u}_i$ we denote the velocity,

$$\dot{u}_i \approx \partial_t u_i \approx v_i \tag{51}$$

We observe that

$$\partial_i \tau_{ij} v_j = \partial_i(\tau_{ij} v_j) - \tau_{(ij)} v_{(j,i)} - \tau_{[ij]} v_{[j,i]} \tag{52}$$

where

$$v_{(j,i)} = \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) \quad , \quad v_{[j,i]} = \dot{\omega}_{ij} = \frac{1}{2}\left(\partial_i \dot{u}_j - \partial_j \dot{u}_i\right) \tag{53}$$

where, using the Eq. (47.2-4) and (47.8), we get:

$$\begin{aligned} \partial_i \tau_{ij} v_j &= \partial_i(\tau_{ij} v_j) - (\hat{\tau}_{ij} - \partial_k \mu_{kij}) v_{(j,i)} + \tfrac{1}{2} e_{ijk} \partial_l \mu_{lk} v_{[j,i]} \\ &= \partial_i(\tau_{ij} v_j) - (\hat{\tau}_{ij} - \underline{\partial_k \mu_{kij}) \dot{\varepsilon}_{ij} + \partial_l \mu_{lk} \dot{w}_k} \end{aligned} \tag{54}$$

Also we have

$$A = \partial_l \mu_{lk} \dot{w}_k + \partial_k \mu_{kij} \dot{\varepsilon}_{ij} = \partial_i \left(\mu_{ij} \dot{w}_j + \mu_{ijk} \dot{\varepsilon}_{jk}\right) \underline{- \mu_{ij} \partial_i \dot{w}_j} - \mu_{ijk} \partial_i \dot{\varepsilon}_{jk} \tag{55}$$

and

$$B = \mu_{ij} \partial_i \dot{w}_j = \mu_{ij} \varepsilon_{jlk} \dot{\hat{\kappa}}_{lik} = \frac{1}{2} \mu_{ij} \varepsilon_{jlk} \dot{\hat{\kappa}}_{lik} + \frac{1}{2} \mu_{kj} \varepsilon_{jli} \dot{\hat{\kappa}}_{lki} \tag{56}$$

Eq. (55) becomes

$$
\begin{aligned}
A &= \partial_i\left(\mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right) - B - \mu_{ijk}\dot{\hat{\kappa}}_{ijk} \\
&= \partial_i\left(\mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right) - \left(\mu_{ijk} + \frac{1}{2}\varepsilon_{lij}\mu_{kl} + \frac{1}{2}\varepsilon_{lik}\mu_{jl}\right)\dot{\hat{\kappa}}_{ijk} \\
&= \partial_i\left(\mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right) - \hat{\mu}_{ijk}\dot{\hat{\kappa}}_{ijk}
\end{aligned}
\tag{57}
$$

where Eq. (29)[8] was used,

$$
\hat{\mu}_{ijk} = \mu_{ijk} + \frac{1}{2}\varepsilon_{lik}\mu_{jl} + \frac{1}{2}\varepsilon_{lij}\mu_{kl} \tag{29}
$$

Substituting Eq. (54) with Eq. (57) into the integrand of Eq. (50.2) and by using Eq. (7.2) for the rate of elastic energy density

$$
\dot{\hat{w}} = \hat{\tau}_{ij}\dot{\varepsilon}_{ij} + \hat{\mu}_{ijk}\dot{\hat{\kappa}}_{ijk} \tag{7.2}
$$

we get

$$
\begin{aligned}
\Pi &= F_j v_j + \partial_i(\tau_{ij}v_j) - \left(\hat{\tau}_{jk}\dot{\varepsilon}_{jk} + \hat{\mu}_{ijk}\dot{\hat{\kappa}}_{ijk}\right) + \partial_i\left(\mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right) \\
&= F_j v_j + \partial_i(\tau_{ij}v_j + \mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}) - \dot{\hat{w}}
\end{aligned}
\tag{58}
$$

Using now the divergence theorem we get

$$
\int_V \Pi \, dV = 0 \quad \Rightarrow \quad \int_V \dot{\hat{w}} dV = \int_V (F_k v_k) dV + \int_{\partial V}\left(\tau_{ij}v_j + \mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right)n_i \, dS \tag{59}
$$

Thus starting with the system of 71 equations we recovered the energy balance equation for the rates. By integrating over time we get the energy balance equation for the solution vector X,

$$
\int_0^t dt \int_V \dot{\hat{w}} dV = \int_0^t dt \int_V (F_k v_k) dV + \int_0^t dt \int_{\partial V}\left(\tau_{ij}v_j + \mu_{ij}\dot{w}_j + \mu_{ijk}\dot{\varepsilon}_{jk}\right)n_i \, dS \tag{60}
$$

Since we have assumed that there the two different solutions hold for the same system of external forces, then

$$
F_k = F_k' - F_k'' = 0 \tag{61}
$$

and the volume integral in Eq.(60) vanishes.

---

[8] Appendix I

We remark that in the energy balance Eq. (60) appear the velocity v_i and its derivatives $\partial_i v_j$, through the definitions of the spin vector and the strain-rate tensor, respectively. It can be shown that <u>not</u> all partial derivatives of a field prescribed along a boundary can be chosen independently[9]. Actually, if along a smooth boundary a component of the velocity field v_i is given, then at the same boundary <u>only</u> its normal derivative Dv_i, with $D = n_k \partial_k$ may be independently prescribed. Thus starting with the system of 71 equations we get the equation of virtual work for the difference solution vector X. We have to reduce the 12 dependent variables of the surface integral in Eq. (60) to 6, because only 6 are independent. So, if it is assumed that the velocity v_k and its normal derivative Dv_k are known on the boundary ∂V we write

$$\int_{\partial V} \left(\tau_{ij} v_j + \mu_{ij} \dot{w}_j + \mu_{ijk} \dot{\varepsilon}_{jk} \right) n_i \, dS = \int_{\partial V} \left(P_k v_k + Q_k D v_k \right) dS \tag{62}$$

In Appendix III we show that indeed for the problem, as is formulated above by Eqs. (47), the forcing vectors P_k and Q_k may be expressed in terms of the constitutive stresses and the mean torsion as follows:

$$P_k = \left(\left(\hat{\tau}_{jk} - \partial_i \left(\hat{\mu}^t{}_{ijk} - \hat{\mu}^t{}_{jik} \right) - n_l n_m \partial_m \hat{\mu}^t{}_{jlk} \right) - \left(\hat{\mu}^t{}_{jik} - n_m n_n \delta_{ij} \hat{\mu}^t{}_{mnk} + \frac{1}{2} \varepsilon_{jik} \mu \right) D_i \right) n_j \tag{63}$$

$$Q_k = \hat{\mu}^t{}_{ijk} \, n_i n_j \tag{64}$$

where

$$D = n_k \partial_k \quad , \quad D_k = \partial_k - n_k D \tag{65}$$

and

$$\hat{\mu}^t{}_{ijk} = \frac{1}{3} (\hat{\mu}_{ijk} + 2\hat{\mu}_{jki}) \tag{66}$$

Finally from Eqs. (60) and (62) we get

---

[9] Appendix IV

$$\int_V \hat{w}dV = \int_0^t dt \int_{\partial V_u} (P_k v_k + Q_k Dv_k) dS + \int_{\partial V_\sigma} (P_k v_k + Q_k Dv_k) dS$$
$$= \frac{1}{2}\left\{ \int_{\partial V_u} (P_k u_k + Q_k Du_k) dS + \int_{\partial V_\sigma} (P_k u_k + Q_k Du_k) dS \right\} \tag{67}$$

where ∂V_u is that subset of the boundary, where kinematic constraints are given, and ∂V_σ is the subset, where static constraints are given. However we have that,

- on $\partial V_u : u_k = u'_k - u''_k = 0$, $Du_k = Du'_k - Du''_k = 0$ (68)

- on ∂V_σ: $P_k = P'_k - P''_k = 0$, $Q_k = Q'_k - Q''_k = 0$ (69)

and we get finally that

$$\int_V \hat{w}dV = 0 \tag{70}$$

The elastic strain energy-density function has been proven to be positive as soon as the following constitutive inequalities hold (Georgiadis et al. 2002),

$$(3\lambda + 2G) > 0 , \ G > 0 \ , \ \ell^2 > 0 \ , \ -1 < \frac{\ell'}{\ell} < 1 \tag{71}$$

Thus, considering Eq. (2), the only solution that Eq. (60) admits is the trivial one,

$$\varepsilon_{ij} = 0 \ \text{ and } \ \hat{\kappa}_{ijk} = \partial_i \varepsilon_{jk} = 0 \tag{72}$$

If rigid body motions are excluded from the boundary constraints, then this results to

$$u_i = u'_i - u''_i = 0 \tag{73}$$

which proves that the two solutions are identical.

We remark that the constitutive stresses $\hat{\tau}_{jk}$, $\hat{\mu}_{ijk}$ can be computed exactly, whereas the true stresses $\tau_{jk}, \mu_{ij}, \mu_{ijk}$ can be computed only within an approximation of the mean torsion. We notice however that the value of the mean torsion does not influence the uniqueness of the solution of the considered boundary-value problem. Thus, as already mentioned above, we will assume that the spherical part of the couple-stress tensor is vanishing ($\mu = 0$); cf. Eq. (42).

3.6 Boundary Conditions

Let us consider a micro-elastic body **B,** with volume V and boundary ∂V. In a classical continuum, the boundary is divided in two complementary parts s.t.: $\partial V = \partial V_u \cup \partial V_\sigma$, $\partial V_u \cap \partial V_\sigma = \varnothing$. The set ∂V_u is that set, where displacements u_i are prescribed; ∂V_σ is the complementary set, where tractions t_i are prescribed, such that at any point $x_k \in \partial V$ $u_i t_i = 0$. In a higher grade continuum, like the one we are studying here, in addition to stresses and displacements, higher order stresses and displacement derivatives are needed to be known on the boundary. In particular from Eq. (62) we get the virtual work of the external forces on the boundary ∂V_σ, where external forces are prescribed. By forcing the virtual displacement and its normal derivative to vanish on $\partial V_u = \partial V - \partial V_\sigma$, we get

$$\delta W_S^{ext} = \int_{\partial V} \left(P_k \delta u_k + Q_k D \delta u_k \right) n_i \, dS \tag{74}$$

In this expression P_k and Q_k are vectors, which work on the displacement and on its normal derivative. As shown in Sect. 3.4 the forcing vectors depend on the constitutive stresses according to Eqs. (63) and (64). For easy reference we summarize here these conditions, considering the assumption of vanishing mean torsion:

$$\begin{aligned}
&P_k = \left(\left(\hat{\tau}_{jk} - \partial_i \left(\hat{\mu}^t{}_{ijk} - \hat{\mu}^t{}_{jik} \right) - n_l n_m \partial_m \hat{\mu}^t{}_{jlk} \right) - \left(\hat{\mu}^t{}_{jik} - n_m n_n \delta_{ij} \hat{\mu}^t{}_{mnk} \right) D_i \right) n_j \\
&Q_k = \hat{\mu}^t{}_{ijk} \, n_i n_j \\
&D = n_k \partial_k \quad , \quad D_k = \partial_k - n_k D \\
&\hat{\mu}^t{}_{ijk} = \frac{1}{3} (\hat{\mu}_{ijk} + 2\hat{\mu}_{jki}) \\
&\hat{\tau}_{ij} = \frac{\partial \hat{w}}{\partial \varepsilon_{ij}} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij} + \ell'_k \left(\lambda \hat{\kappa}_{knn} \delta_{ij} + 2G\hat{\kappa}_{kij} \right) \\
&\hat{\mu}_{ijk} = \frac{\partial \hat{w}}{\partial \hat{\kappa}_{ijk}} = \ell^2 \left(\lambda \hat{\kappa}_{inn} \delta_{jk} + 2G\hat{\kappa}_{ijk} \right) + \ell'_i \left(\lambda \varepsilon_{nn} \delta_{jk} + 2G\varepsilon_{jk} \right)
\end{aligned} \tag{75}$$

From the virtual work Eq. (74) follows that the vector P_k has clear physical meaning: P_k is the <u>total</u>, work-producing, boundary-traction; i.e. of the total force that does work on the boundary displacement. We will call P_k the *working traction or the potency*. We remark that the potency P_k differs from both traction vectors that can be build from the constitutive or the true stress tensors

$$P_k \neq \hat{\tau}_{ik} n_i \quad , \quad P_k \neq \tau_{ik} n_i \tag{76}$$

In that sense we use the potency P_k for defining boundary conditions for tractions, by identifying the notion of "force" with the notion of "work producing force". The other work producing quantity is the vector Q_k. Its physical meaning is less clear, but we will try to shed some light to it by working out a simple example.

Thus we conclude that at any boundary point of the considered micro-elastic body **B** and for the directions normal to- and in the tangential plane we can prescribe either static quantities or their energy conjugate kinematic quantities, by selecting from the set[10] $\{P_k, Q_k \mid u_k, Du_k\}$ such that at any point $x_k \in \partial V \;\; u_i P_i = 0 \wedge Du_i Q_i = 0$.

3.7 Examples

The uniqueness theorem, presented in section 3.5, justifies among others the use of the so-called *"semi-inverse method"* of solving b.v. problems. According to this method a strain field with some general properties is selected before hand and then a solution is constructed, which fulfills all field and constitutive equations as well as sufficient boundary conditions. Then according to the above theorem such a solution must be unique.

3.7.1 One-dimensional Compression

As a first application of the present theory we will consider here the problem of one-dimensional compression[11] (Fig. 4) of a material specimen of initial height H.

The assumed strain field is,

$$[\varepsilon] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \;, \;\; \varepsilon_{33} = \hat{\varepsilon}(x_3) = \frac{du_3}{dx_3} \tag{77}$$

Thus the only non-vanishing component of the strain gradient is,

$$\hat{\kappa}_{333} = \frac{d\hat{\varepsilon}}{dx_3} \tag{78}$$

---

[10] As is known from Continuum Mechanics, other boundary conditions may be also conceived, like for example the "elastic support", where the traction at a point is a linear function of the displacement of that point, $P_k = ku_k$.

[11] In Soil Mechanics this deformation is called an oedometric compression.

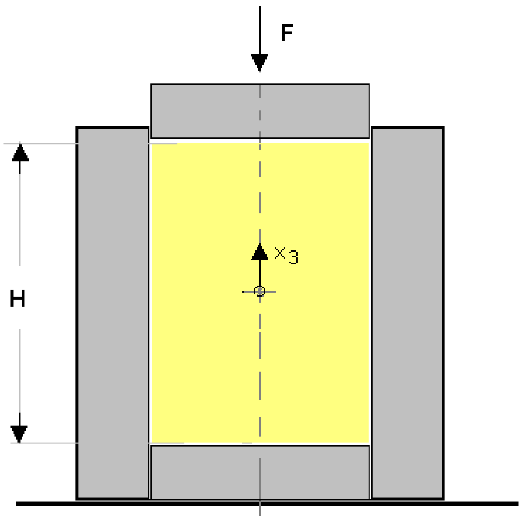

Figure 4. One-dimensional compression

By neglecting the influence of the self weight, the equilibrium Eq. (37) reads

$$\frac{d\sigma_{33}}{dx_3} = 0 \quad \Rightarrow \quad \sigma_{33} = \text{const.} \tag{79}$$

From Eq. (37) we get

$$\sigma_{33} = (\lambda + 2G)\left(1 - \ell^2 \frac{d^2}{dx_3^2}\right)\varepsilon_{33} \tag{80}$$

Thus the equilibrium Eq. (79) yields that

$$\frac{d}{dx_3}\left(\varepsilon_{33} - \ell^2 \frac{d^2\varepsilon_{33}}{dx_3^2}\right) = 0 \tag{81}$$

From that we get the following general solution for the strain-field:

$$\varepsilon_{33} = \varepsilon_0 + c_1 \sinh\left(\frac{x_3}{\ell}\right) + c_2 \cosh\left(\frac{x_3}{\ell}\right) \tag{82}$$

We assume that the boundary potency is known; i.e.

$$\text{at} \quad x_3 = \pm H/2 \; (n_3 = \pm 1): \quad P_1 = P_2 = 0 \quad , \quad P_3 = -\bar{\sigma} n_3 \tag{83}$$

where $\bar{\sigma}$ is computed from the total applied load F and from the cross-sectional area of a horizontal section of the specimen (Fig. 4),

$$\bar{\sigma} = \frac{F}{A} \tag{84}$$

The boundary condition for the potency P_3, Eq. (75.1), yields:

$$P_3 = \hat{\tau}_{33} - \frac{d\hat{\mu}}{dx_3} = \sigma_{33} \quad \Rightarrow \varepsilon_0 = -\frac{\bar{\sigma}}{\lambda + 2G} < 0 \quad (\textit{compressive primary strain}) \tag{85}$$

For solving the problem at hand we must also invoke an additional boundary condition, which will allow for the determination of the remaining integration constants c_1 and c_2 of the strain field, Eq. (82). As explained in the previous section this additional boundary condition could be a statement either for the value of the normal derivative of the vertical displacement (i.e. for the strain ε_{33}) or for the value of the vertical component Q_3 of the corresponding forcing vector. Here we will select the former. This selection is justified as follows: We assume that the specimen is compressed by two “rigid” pistons. Within the classical continuum theory we would admit that at the interface, between specimen and rigid pistons, a weak displacement discontinuity develops, that will allow for the normal to the interface strain, inside the deformable body, to take immediately the value ε_0. In the contrary, within the enhanced continuum theory such an interface does not need to be a strain discontinuity line for the normal strain. Continuity of strain may be achieved by imposing the normal strain to be zero at that interface; i.e.

$$\text{at} \quad x_3 = \pm H/2: \quad Du_3 = \varepsilon_{33} = 0 \quad \Rightarrow \tag{86}$$

$$\begin{array}{lcl} c_1 \sinh\left(\dfrac{H}{2\ell}\right) + c_2 \cosh\left(\dfrac{H}{2\ell}\right) = -\varepsilon_0 & & c_1 = 0 \\ & \Rightarrow & \\ -c_1 \sinh\left(\dfrac{H}{2\ell}\right) + c_2 \cosh\left(\dfrac{H}{2\ell}\right) = -\varepsilon_0 & & c_2 = -\dfrac{\varepsilon_0}{\cosh\left(\dfrac{H}{2\ell}\right)} \end{array} \tag{87}$$

With these values for the integration constants we get from Eq. (72) the following solution (Fig. 5)

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\cosh\left(z^* \frac{H}{\ell} \right)}{\cosh\left(\frac{1}{2} \frac{H}{\ell} \right)} \right) \quad , \quad z^* = \frac{x_3}{H} \in [-\frac{1}{2}, \frac{1}{2}] \tag{88}$$

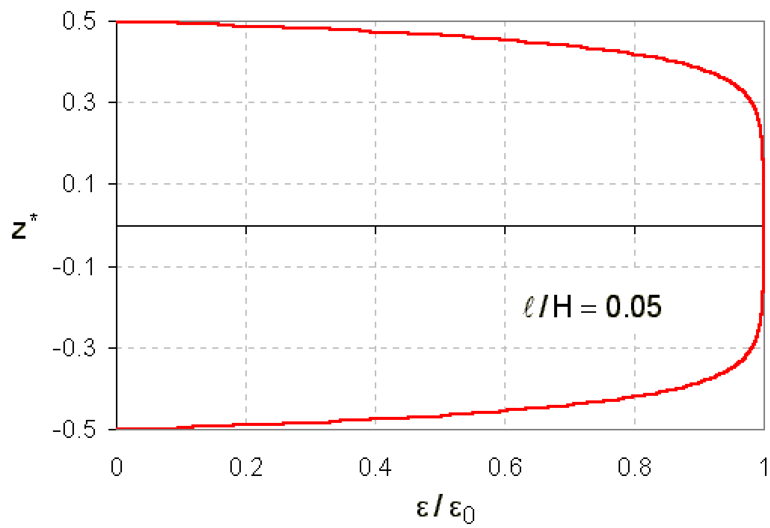

Figure 5. Strain distribution for $\ell / H = 0.05$

We remark that the strain-field, Eq. (88), can be decomposed into two terms as

$$\varepsilon = \varepsilon^{(0)} + \varepsilon^{(2)} \tag{89}$$

where $\varepsilon^{(0)}$ and $\varepsilon^{(2)}$ denote the part of the solution that is due o the 0^{th}- (classical-continuum) and the 2^{nd}-grade (enhanced continuum) terms of the underlying constitutive theory,

$$\varepsilon^{(0)} = \varepsilon_0 = \text{const} \quad , \quad \varepsilon^{(2)} = -\frac{\cosh\left(\frac{H}{\ell} z^* \right)}{\cosh\left(\frac{1}{2} \frac{H}{\ell} \right)} \varepsilon_0 \tag{90}$$

The 2nd grade term $\varepsilon^{(2)}$ perturbs significantly the classical constant-strain solution near to the boundaries. This can be seen easily, if we introduce the transformation,

$$\bar{x}_3 = x_3 - \frac{H}{2} \quad \Rightarrow \quad \bar{z} = \frac{\bar{x}_3}{H} = z^* - \frac{1}{2} \tag{91}$$

Then, for example close to the upper interface boundary, we get that the perturbed strain-field is varying quadratically

$$\varepsilon \approx 2\varepsilon_0 \, e^{-\frac{1}{2}\frac{H}{\ell}} \left(\frac{\bar{x}_3}{\ell}\right)^2 \tag{92}$$

The corresponding displacement field is also a linear combination of a 0th and a 2nd grade term (Fig. 6),

$$u_3 = u_3^{(0)} + u_3^{(2)} \tag{93}$$

where

$$u_3^{(0)} = \varepsilon_0 \left(\frac{1}{2} + z^*\right) H \tag{94}$$

$$u_3^{(2)} = -\varepsilon_0 \left(\tanh\left(\frac{1}{2}\frac{H}{\ell}\right) + \frac{\sinh\left(\frac{H}{\ell} z^*\right)}{\cosh\left(\frac{1}{2}\frac{H}{\ell}\right)} \right) \ell \tag{95}$$

For a given load the compliance of the specimen can be expressed in terms of the engineering strain,

$$\bar{\varepsilon} = \frac{\Delta H}{H} = \frac{u_3(H)}{H} \tag{96}$$

In case of classical elasticity we get,

$$\bar{\varepsilon}^{(0)} = \frac{u_3^{(0)}}{H} = -\varepsilon_0 = \frac{F/A}{\lambda + 2G} \tag{97}$$

Thus micro-elasticity increases the compliance of the specimen, due to the extra terms in the strain energy-density function. According to Eq. (98) the scale effect is linear,

$$\bar{\varepsilon} \approx \bar{\varepsilon}^{(0)}\left(1+2\left(\frac{\ell}{H}\right)\right) \quad \text{for}: \frac{H}{\ell} >> 1 \tag{98}$$

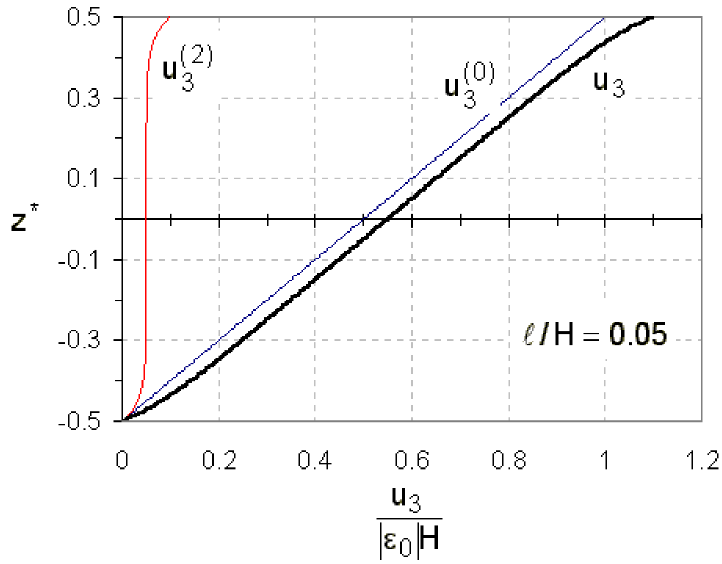

Figure 6. Vertical displacement distribution for, $\ell / L = 0.05$. The linear term, the non-linear (boundary-layer) solution and their normalized superposition, indicating the increased compliance of the struct*ure due to the extra strain-gradient energy stored.*

From the above solutions, Eqs. (89) and (93), we obtain that the effect of the micro-structure is felled only close to the boundaries and that the micro-elasticity model allows for the formation of a boundary layer, within which a gradual adjustment is achieved of the normal strain from its zero value at the interface to its steady-value ε_0 in some distance from it. The internal length ℓ determines the "thickness" of the boundary layer; i.e. the material constant ℓ determines how close to the boundary the perturbation of the far-field strain is still significant. In the present example the director field ℓ'_k does not affect the solution.

Remark

From the above derivation we can see clearly that the present micro-elasticity theory may be used to regularize the discontinuous solutions arising for example from the classical continuum formulation of problems in layered media. In these problems, classical continuum approach can only assure continuity of displacements and their derivatives parallel to the interface, a condition known as Maxwell's compatibility condition (see Vardoulakis & Sulem, 1995). In the case of micro-elastic media, however, we can always assure a smoother interface condition by requiring the continuity of the normal strain as well,

$$[v_i] = v_i^+ - v_i^- = 0 \quad \textit{(continuity of displacement)}$$

$$[\partial_i v_j] = 0 \quad \textit{(continuity of strain)}$$

3.7.2 The "Bolted" Layer

We consider an infinitely long layer of thickness H. We neglect body forces and assume that the surfaces of the layer are traction-free. However, we assume that the surfaces are "bolted" by a series of dense double-headed nails, as shown in Fig. 7. The action of the pre-stressed bolts consists in the application of a co-linear force doublet $\{\vec{F}, -\vec{F}\}$, acting over a short distance $\tilde{\ell}$. During the pre-stressing of the bolts, the material in the vicinity of the bolted layers is strained. The work done by the bolts is,

$$W^{ext} = -p\frac{\Delta\tilde{\ell}}{2} - p\frac{\Delta\tilde{\ell}}{2} = -p\frac{1}{2}\varepsilon_{33}\tilde{\ell} - p\frac{1}{2}\varepsilon_{33}\tilde{\ell} = -p\tilde{\ell}\varepsilon_{33} \tag{99}$$

where p is the equivalent pressure, computed from the bolt forces distributed over the surface of the layer.

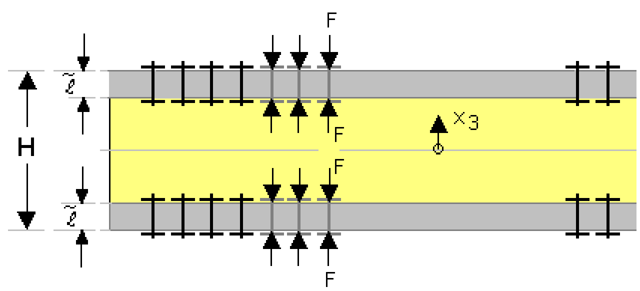

Figure 7. The bolted layer

Thus we may interpret the boundary condition for the forcing vector Q_k, Eq. (75.2) as follows:

$$\begin{aligned} &\text{at } x_3 = H/2 \; (n_3 = +1): \quad Q_3 = \hat{\mu}_{333} = (\lambda + 2G)(\ell^2 \hat{\kappa}_{333} + \ell'_3 (H/2) \varepsilon_{33}) \\ &W_{ext} = Q_3 D u_3 = Q_3 n_3 \partial_3 u_3 = Q_3 \varepsilon_{33} \end{aligned} \tag{100}$$

$$\begin{aligned} &\text{at } x_3 = -H/2 \; (n_3 = -1): \quad Q_3 = \hat{\mu}_{333} = (\lambda + 2G)(\ell^2 \hat{\kappa}_{333} + \ell'_3 (-H/2) \varepsilon_{33}) \\ &W_{ext} = Q_3 D u_3 = Q_3 n_3 \partial_3 u_3 = -Q_3 \varepsilon_{33} \end{aligned} \tag{101}$$

resulting to

$$x_3 = H/2: \quad (\lambda + 2G)(\ell^2 \hat{\kappa}_{333} + \ell'_3 (H/2) \varepsilon_{33}) = -p \tilde{\ell} \tag{102}$$

$$x_3 = -H/2: \quad (\lambda + 2G)(\ell^2 \hat{\kappa}_{333} + \ell'_3 (-H/2) \varepsilon_{33}) = p \tilde{\ell} \tag{103}$$

The considered example is typical of a boundary-value problem with two significant, but otherwise fully identical boundaries. This fact (i.e. the strip-like geometry) excludes the possibility of assuming a constant director field inside the volume of the body (Fig. 8a), because this model would distinguish among the upper and lower surface of the layer. Moreover, since for an observer siting on one of these boundaries there is no reason to distinguish between right and left, we must assume that at boundary points the director field is always directed normal to it (Vardoulakis & Sulem, 1995);

$$\ell'_k = \ell' n_k \quad \forall x_i \in \partial V \tag{104}$$

where n_k is the unit outward normal vector at the boundary[12]. Thus we are forced to assume that the director field is confined into two boundary layers, and that it is having the direction of the outward normal to the corresponding boundary (Fig. 8b).

[12] Of course, we could equally assume that the director is pointing inwards; this is taken care, however, by the sign of ℓ'.

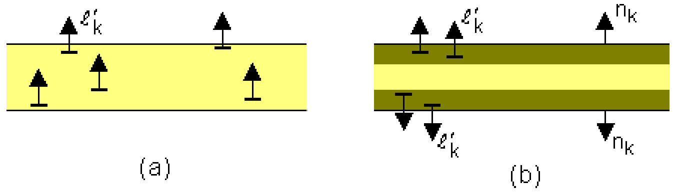

Figure 8. (a) Constant director field (b) Piece-wise constant director field

With these remarks from Eqs (102) to (104) we get,

$$x_3 = H/2: \quad \ell \frac{d\varepsilon_{33}}{dx_3} + \frac{\ell'}{\ell}\varepsilon_{33} = -r \tag{105}$$

$$x_3 = -H/2: \quad \ell \frac{d\varepsilon_{33}}{dx_3} - \frac{\ell'}{\ell}\varepsilon_{33} = r \tag{106}$$

where r is the dimensionless double force, applied at the boundary, scaled by the elastic oedometric modulus and the material length ℓ,

$$r = \frac{p}{\lambda + 2G}\frac{\tilde{\ell}}{\ell} \tag{107}$$

With the general solution, Eq. (82), and the observation that in this case $P_3 = 0\ (\varepsilon_0 = 0)$, from Eqs. (105) and (106) we get the following solution for the strain field,

$$\varepsilon_{33} \approx -\frac{r}{1+\left(\frac{\ell'}{\ell}\right)} \frac{\cosh\left(\frac{x_3}{H}\frac{H}{\ell}\right)}{\cosh\left(\frac{1}{2}\frac{H}{\ell}\right)} \qquad \text{for}: \frac{H}{\ell} >> 1 \tag{108}$$

As can be seen from Fig. 9, the strain is compressive and clearly localized close to the boundaries.

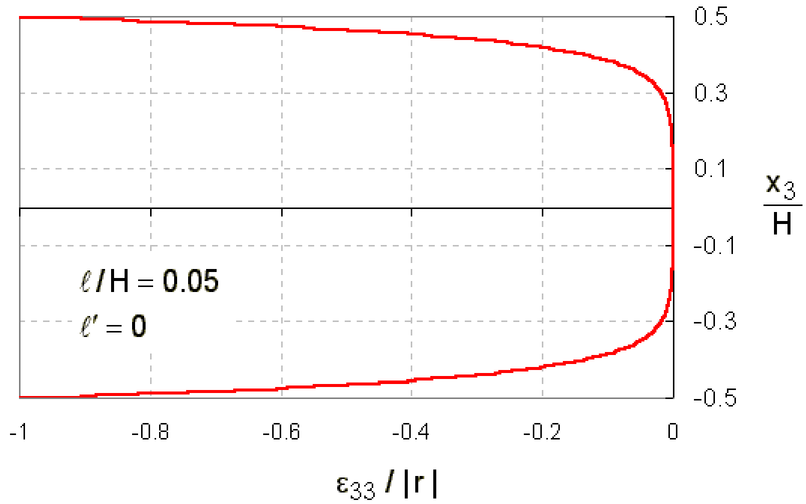

Figure 9. Strain distribution in a bolted layer

We remark that the net relative compression of the layer due to bolting is,

$$\bar{\varepsilon} = -\frac{1}{H}\int_{-H/2}^{H/2} \varepsilon_{33} dx_3 \approx 2\frac{p}{\lambda + 2G}\frac{\left(\frac{\tilde{\ell}}{H}\right)}{1+\left(\frac{\ell'}{\ell}\right)} \quad \text{for} : \frac{H}{\ell} >> 1 \tag{109}$$

It is remarkable that the scale effect is expressed directly by the bold length, which enters into the problem through the boundary condition for double forces. The material lengths appear only through their ratio. According to Eqs. (108) and (109) the effect of the bolts is reduced or amplified depending on the magnitude and sign of the director length ℓ'. As already mentioned, meaningful, within the frame of the present micro-elasticity theory, are values: $-1 < \frac{\ell'}{\ell} < 1$; cf. Ineq. (71). Above example, indicates that most probably negative values of ℓ' should rather be excluded, since they lead to infinite compliance for limit $\ell' / \ell \to -1$.

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Appendix I : *Relations between true and constitutive stresses*

Energy balance is expressed by the equation between the total elastic energy rate and the power of external forces,

$$\underline{\dot{W}^{el} = W^{ext}} \tag{A1.1}$$

Using Eq. (7.2) for the elastic strain energy density function we get

$$\dot{W}^{el} = \int_V (\hat{\tau}_{jk}\dot{\varepsilon}_{jk} + \hat{\mu}_{lik}\dot{\hat{\kappa}}_{lik})dV \tag{A1.2}$$

Using the divergence theorem and the definitions of the kinematic variables, Eqs. (3), the power of external forces becomes:

$$W^{(e)} = \int_V \left\{ (F_j + \partial_i\tau_{ij})v_j + \tau_{ij}\partial_i v_j + \partial_i\mu_{ij}\dot{w}_j + \mu_{ij}\partial_i\dot{w}_j + \partial_i\mu_{ijk}\dot{\varepsilon}_{jk} + \mu_{ijk}\partial_i\dot{\varepsilon}_{jk} \right\} dV \tag{A1.3}$$

We remark that:

1. $\underline{F_j + \partial_i\tau_{ij} = 0}$

due to the equilibrium Eq. (13), and

2. $\tau_{ij}\partial_i v_j = \tau_{(ij)}\dot{\varepsilon}_{ij} + \tau_{[ij]}\dot{w}_{ij} = \tau_{(ij)}\dot{\varepsilon}_{ij} + \tau_{[ij]}e_{jki}\dot{w}_k$ (A1.4)

3. $\mu_{ij}\partial_i\dot{w}_j = \mu_{ij}\varepsilon_{jlk}\dot{\hat{\kappa}}_{lik} = \frac{1}{2}\mu_{ij}\varepsilon_{jlk}\dot{\hat{\kappa}}_{lik} + \frac{1}{2}\mu_{kj}\varepsilon_{jli}\dot{\hat{\kappa}}_{lki}$ (A1.5)

4. $\mu_{ijk}\partial_i\dot{\varepsilon}_{jk} = \mu_{ijk}\dot{\hat{\kappa}}_{ijk} = \mu_{lik}\dot{\hat{\kappa}}_{lik}$ (A1.6)

Introducing the above expressions into Eq. (A1.3) and using Eqs. (A1.1) and (A1.2) we get the following equation:

$$
\begin{aligned}
&\int_V (\hat{\tau}_{jk}\dot{\varepsilon}_{jk} + \hat{\mu}_{lik}\dot{\kappa}_{lik})dV = \\
&\int_V \left\{ (\tau_{[ki]}\varepsilon_{ijk} + \partial_i\mu_{ij})\dot{w}_j + (\tau_{(jk)} + \partial_i\mu_{ijk})\dot{\varepsilon}_{jk} + (\frac{1}{2}\mu_{ij}\varepsilon_{jlk} + \frac{1}{2}\mu_{kj}\varepsilon_{jli} + \mu_{lik})\dot{\kappa}_{lik} \right\} dV
\end{aligned}
\tag{A1.7}
$$

We remark that:

5. $\underline{\partial_j\mu_{jk} + \varepsilon_{ijk}\tau_{ij} = 0}$, $\quad \Rightarrow \quad \partial_i\mu_{ij} + \varepsilon_{ijk}\tau_{[ki]} = 0 \quad (\varepsilon_{ijk}\tau_{(ki)} = 0)$

due to the moment equilibrium Eq. (16), and therefore Eq. (A1.7) becomes

$$
\begin{aligned}
&\int_V (\hat{\tau}_{jk}\dot{\varepsilon}_{jk} + \hat{\mu}_{lik}\dot{\kappa}_{lik})dV = \\
&\int_V \left\{ (\tau_{(jk)} + \partial_i\mu_{ijk})\dot{\varepsilon}_{jk} + (\frac{1}{2}\mu_{ij}\varepsilon_{jlk} + \frac{1}{2}\mu_{kj}\varepsilon_{jli} + \mu_{lik})\dot{\kappa}_{lik} \right\} dV
\end{aligned}
\tag{A1.8}
$$

We restrict first the kinematics to the particular class of strain-rate fields which are homogeneous inside a volume $V' \subseteq V$ (i.e. strain fields with $\dot{\kappa}_{lik} = \partial_l\dot{\varepsilon}_{ik} = 0$) and which vanish outside V'. If we apply Eq. (A1.8) for these fields then,

$$
\int_{V'} \left\{ \hat{\tau}_{jk} - (\tau_{(jk)} + \partial_i\mu_{ijk}) \right\} \dot{\varepsilon}_{jk})dV = 0 \quad \Rightarrow \quad \underline{\hat{\tau}_{jk} = \tau_{(jk)} + \partial_i\mu_{ijk}}
\tag{28}
$$

With this result we get then from Eq. (A1.8) also that

$$
\underline{\hat{\mu}_{ijk} = \mu_{ijk} + \frac{1}{2}\mu_{jl}\varepsilon_{lik} + \frac{1}{2}\mu_{kl}\varepsilon_{lij}}
\tag{29}
$$

Appendix II: Evaluation of the deviatoric part of the true couple stress and of the expression of the true double stress

We start with

$$\mu_{ij}^{D} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{jpq} \tag{31}$$

The proof of Eq. (31) can be found in the paper by M&E.

We consider the 1st form of elastic strain energy density function after M&E

$$w = \widetilde{w}(\varepsilon_{ij}, \widetilde{\kappa}_{ijk}) \quad , \quad \widetilde{\kappa}_{ijk} = \partial_i \partial_j u_k \tag{A2.1}$$

and we define the corresponding constitutive stresses

$$\widetilde{\mu}_{ijk} = \frac{\partial \widetilde{w}}{\partial \widetilde{\kappa}_{ijk}} = \frac{\partial \hat{w}}{\partial \hat{\kappa}_{pqr}} \frac{\partial \hat{\kappa}_{pqr}}{\partial \widetilde{\kappa}_{ijk}} = \hat{\mu}_{pqr} \frac{\partial \hat{\kappa}_{pqr}}{\partial \widetilde{\kappa}_{ijk}} \tag{A2.2}$$

By using the relation of the kinematic variables between the 1st and 2nd form we get

$$\hat{\kappa}_{pqr} = \tfrac{1}{2}(\widetilde{\kappa}_{pqr} + \widetilde{\kappa}_{prq}) = \tfrac{1}{2}\widetilde{\kappa}_{ijk}(\delta_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}\delta_{jr}\delta_{kq}) \tag{A2.3}$$

Then Eq. (A2.2) becomes

$$\widetilde{\mu}_{ijk} = \hat{\mu}_{pqr}\tfrac{1}{2}(\delta_{ip}\delta_{jq}\delta_{kr} + \delta_{ip}\delta_{jr}\delta_{kq}) = \tfrac{1}{2}(\hat{\mu}_{ijk} + \hat{\mu}_{ikj}) \tag{A2.4}$$

We consider also the **3rd form** of the elastic energy density function

$$w = \overline{w}(\varepsilon_{ij}, \overline{\kappa}_{ij}, \overline{\overline{\kappa}}_{ijk}) \tag{A2.5}$$

with,

$$\overline{\kappa}_{ij} = \frac{1}{2}\varepsilon_{jlk}\partial_l \partial_i u_k \tag{A2.6}$$

$$\overline{\overline{\kappa}}_{ijk} = \frac{1}{3}\left(\partial_i \partial_j u_k + \partial_j \partial_k u_i + \partial_k \partial_i u_j\right) \tag{A2.7}$$

and we define the corresponding constitutive stresses

$$\overline{\overline{\mu}}_{ijk} = \frac{\partial \overline{w}}{\partial \overline{\overline{\kappa}}_{ijk}} = \frac{\partial \hat{w}}{\partial \hat{\kappa}_{pqr}} \frac{\partial \hat{\kappa}_{pqr}}{\partial \overline{\overline{\kappa}}_{ijk}} \tag{A2.8}$$

$$\overline{\mu}_{ij} = \frac{\partial \overline{w}}{\partial \overline{\kappa}_{ij}} = \frac{\partial \widetilde{w}}{\partial \widetilde{\kappa}_{pqr}} \frac{\partial \widetilde{\kappa}_{pqr}}{\partial \overline{\kappa}_{ij}} = \widetilde{\mu}_{pqr} \frac{\partial \widetilde{\kappa}_{pqr}}{\partial \overline{\kappa}_{ij}} \tag{A2.9}$$

We are using further the relation between the kinematic variables of the 1st and 3rd energy form

$$\begin{aligned} \widetilde{\kappa}_{pqr} &= \overline{\overline{\kappa}}_{pqr} + \tfrac{2}{3}\overline{\kappa}_{pl}\varepsilon_{lqr} + \tfrac{2}{3}\overline{\kappa}_{ql}\varepsilon_{lpr} = \overline{\overline{\kappa}}_{pqr} + \tfrac{2}{3}\overline{\kappa}_{ij}(\delta_{ip}\delta_{lj}\varepsilon_{lqr} + \delta_{iq}\delta_{lj}\varepsilon_{lpr}) = \\ &= \overline{\overline{\kappa}}_{pqr} + \tfrac{2}{3}\overline{\kappa}_{ij}(\delta_{ip}\varepsilon_{jqr} + \delta_{iq}\varepsilon_{jpr}) \end{aligned} \tag{A2.10}$$

Then Eq. (A2.9) gives

$$\begin{aligned} \overline{\mu}_{ij} &= \frac{2}{3}(\widetilde{\mu}_{iqr}\varepsilon_{jqr} + \widetilde{\mu}_{pir}\varepsilon_{jpr}) = \frac{2}{3}(\widetilde{\mu}_{iqr}\varepsilon_{jqr} + \widetilde{\mu}_{ipr}\varepsilon_{jpr}) \\ &= \frac{4}{3}\widetilde{\mu}_{ipr}\varepsilon_{jpr} = \frac{4}{3}\widetilde{\mu}_{ipq}\varepsilon_{jpq} \end{aligned} \tag{A2.11}$$

From Eqs. (A2.4) and (A2.11) we obtain

$$\overline{\mu}_{ij} = \frac{2}{3}(\hat{\mu}_{ipq} + \hat{\mu}_{piq})\varepsilon_{jpq} \tag{A2.12}$$

and with that and Eq.(31) we have proven Eq. (32)

$$\underline{\overline{\mu}_{ij} = \mu_{ij}^{D}} \tag{32}$$

The energy balance equation for the 3rd energy form reads,

$$W^{e} = W^{el} \quad \Rightarrow \quad \int_V \dot{\overline{w}} dV = \int_V F_k v_k dV + \int_{\partial V} \left(t_k v_k + m_k \dot{w}_k + R_{ij}\dot{\varepsilon}_{ij}\right) dS \tag{A2.13}$$

We remark also the following equations

$$\mu_{ij}\partial_i \dot{w}_j = (\mu_{ij}^{D} + \tfrac{1}{3}\delta_{ij}\mu_{kk})\dot{\overline{\kappa}}_{ij} = \mu_{ij}^{D}\dot{\overline{\kappa}}_{ij} \tag{A2.14}$$

$$
\begin{aligned}
\mu_{ijk}\partial_i\dot{\varepsilon}_{jk} &= \mu_{ijk}\dot{\hat{\kappa}}_{ijk} = \mu_{ijk}(\dot{\bar{\bar{\kappa}}}_{ijk} - \tfrac{1}{3}\dot{\bar{\kappa}}_{jl}\varepsilon_{kil} - \tfrac{1}{3}\dot{\bar{\kappa}}_{kl}\varepsilon_{jil}) \\
&= \mu_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk} - \tfrac{1}{3}\mu_{ijk}(\delta_{pj}\delta_{ql}\varepsilon_{kil} + \delta_{pk}\delta_{ql}\varepsilon_{jil})\dot{\bar{\kappa}}_{pq} \\
&= \mu_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk} - \tfrac{2}{3}\mu_{ipk}\varepsilon_{kiq}\dot{\bar{\kappa}}_{pq}
\end{aligned} \tag{A2.15}
$$

Using the expression of the rate of elastic energy density

$$
\dot{\overline{w}} = \bar{\tau}_{ij}\dot{\varepsilon}_{ij} + \bar{\mu}_{ij}\dot{\bar{\kappa}}_{ij} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk} \tag{A2.16}
$$

and replacing all the above Eqs. (A2.14-16) into Eq. (A2.13) and using the divergence theorem we get:

$$
\begin{aligned}
&\int_V (\bar{\tau}_{jk}\dot{\varepsilon}_{jk} + \bar{\mu}_{pq}\dot{\bar{\kappa}}_{pq} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk})dV = \\
&= \int_V \left\{(F_j + \partial_i\tau_{ij})v_j + \tau_{ij}\partial_i v_j + \partial_i\mu_{ij}\dot{w}_j + \mu_{ij}\partial_i\dot{w}_j + \partial_i\mu_{ijk}\dot{\varepsilon}_{jk} + \mu_{ijk}\partial_i\dot{\varepsilon}_{jk}\right\}dV \\
&= \int_V \left\{(\tau_{[ik]}\varepsilon_{kji} + \partial_i\mu_{ij})\dot{w}_j + (\tau_{(jk)} + \partial_i\mu_{ijk})\dot{\varepsilon}_{jk} + (\mu^D_{pq} - \tfrac{2}{3}\mu_{ipk}\varepsilon_{kiq})\dot{\bar{\kappa}}_{pq} + \mu_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk}\right\}dV
\end{aligned} \tag{A2.17}
$$

From this expression and

$$
\tau_{[ik]} = -\tfrac{1}{2}\varepsilon_{ikm}\partial_l\mu^D_{lm}
$$

Eq. (A2.17) becomes

$$
\begin{aligned}
&\int_V (\bar{\tau}_{jk}\dot{\varepsilon}_{jk} + \bar{\mu}_{pq}\dot{\bar{\kappa}}_{pq} + \bar{\bar{\mu}}_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk})dV = \\
&= \int_V \left\{(\tau_{(jk)} + \partial_i\mu_{ijk})\dot{\varepsilon}_{jk} + (\mu^D_{pq} - \tfrac{2}{3}\mu_{ipk}\varepsilon_{kiq})\dot{\bar{\kappa}}_{pq} + \mu_{ijk}\dot{\bar{\bar{\kappa}}}_{ijk}\right\}dV
\end{aligned} \tag{A2.18}
$$

Using a similar argument as in Appendix I we obtain finally from (A2.18) the following local equations

$$
\bar{\tau}_{jk} = \tau_{(jk)} + \partial_i\mu_{ijk} \tag{A2.19}
$$

$$
\bar{\mu}_{pq} = \mu^D_{pq} - \tfrac{2}{3}\mu_{ipk}e_{kiq} \tag{A2.20}
$$

$$
\bar{\bar{\mu}}_{ijk} = \mu_{ijk} \tag{A2.21}
$$

Finally by using its definition (cf. M&E) we get

$$\overline{\overline{\mu}}_{ijk} = \frac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}) \tag{A2.22}$$

Thus from Eqs. (A2.21) and (A2.22) we get

$$\underline{\mu_{ijk} = \frac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij})} \tag{33}$$

For the proof of Eq. (34) we remark first that[13]

$$\varepsilon_{mnp}\varepsilon^{rst} = \delta^{rst}_{mnp} = \begin{vmatrix} \delta^r_m & \delta^r_n & \delta^r_p \\ \delta^s_m & \delta^s_n & \delta^s_p \\ \delta^t_m & \delta^t_n & \delta^t_p \end{vmatrix} \tag{A2.23}$$

$$\delta_{ij} = \delta^{ij} = \delta^j_i = \begin{cases} 1 & \text{if} : i = j \\ \\ 0 & \text{if} : i \neq j \end{cases} \quad , \quad \varepsilon_{ijk} = \varepsilon^{ijk} \quad \text{etc.} \tag{A2.24}$$

In particular we get

$$\delta^{rsp}_{mnp} = \delta^{rs}_{mn} = \begin{vmatrix} \delta^r_m & \delta^r_n \\ \delta^s_m & \delta^s_n \end{vmatrix} \tag{A2.25}$$

With Eq. (27), Eq. (29) becomes

$$\mu^D_{il}\varepsilon_{ljk} + \mu^D_{jl}\varepsilon_{lki} + \mu^D_{kl}\varepsilon_{lij} = 0 \tag{A2.26}$$

where μ^D_{ij} is given by Eq. (31). Thus by using Eq. (26), (A2.23-5) we conclude by inspection that Eq. (A2.26) is correct.

[13] McConnell A.J. *Applications of Tensor Analysis,* Dover, 1957.

Appendix III: On the uniqueness theorem

We consider the virtual work equation

$$\int_{\partial V}\left(\tau_{ij}\delta u_j + \mu_{ij}\delta w_j + \mu_{ijk}\delta\varepsilon_{jk}\right)n_i\, dS = \int_{\partial V}\left(P_k\delta u_k + Q_k D\delta u_k\right)dS \tag{A3.1}$$

- The first term of the left hand side of Eq. (A3.1) becomes:

$$\begin{aligned}\tau_{jk} &= \tau_{(jk)} + \tau_{[jk]} = \hat{\tau}_{jk} - \partial_i\mu_{ijk} - \frac{1}{2}\varepsilon_{ijk}\partial_l\mu_{li} = \\ &= \hat{\tau}_{jk} - \frac{1}{3}\partial_i\left(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}\right) - \frac{1}{2}\varepsilon_{ijk}\partial_l\mu_{li}^D - \frac{1}{2}\varepsilon_{ijk}\delta_{li}\partial_l\mu\end{aligned} \tag{A3.2}$$

where μ is the mean torsion

$$\mu = \tfrac{1}{3}\mu_{nn} \tag{A3.3}$$

Using Eq. (47.9)

$$\begin{aligned}\tau_{jk} &= \hat{\tau}_{jk} - \frac{1}{3}\partial_i\left(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}\right) - \frac{1}{3}\partial_l\left(\hat{\mu}_{jkl} - \hat{\mu}_{kjl}\right) - \frac{1}{2}\varepsilon_{ijk}\delta_{li}\partial_l\mu \\ &= \hat{\tau}_{jk} - \frac{1}{3}\partial_i\left(\hat{\mu}_{ijk} + 2\hat{\mu}_{jki}\right) - \frac{1}{2}\varepsilon_{ijk}\delta_{li}\partial_l\mu\end{aligned} \tag{A3.4}$$

and therefore

$$\underline{\int_{\partial V} n_j\tau_{jk}\delta u_k\, dS = \int_{\partial V} n_j\left\{\hat{\tau}_{jk} - \frac{1}{3}\partial_i\left(\hat{\mu}_{ijk} + 2\hat{\mu}_{jki}\right) - \frac{1}{2}\varepsilon_{ijk}\delta_{li}\partial_l\mu\right\}\delta u_k\, dS} \tag{A3.5}$$

- The second term of the left hand side of Eq. (A3.1) becomes:

$$n_i\mu_{ij}\delta w_j = n_i\mu_{ij}^D\delta w_j + n_i\mu\delta_{ij}\delta w_j \tag{A3.6}$$

Analyzing the displacement gradient into normal and tangential derivative

$$w_j = \frac{1}{2}\varepsilon_{jkl}\partial_k\delta u_l = \frac{1}{2}\varepsilon_{jkl}\left(n_k D\delta u_l + D_k\delta u_l\right) \tag{A3.7}$$

we get:

$$n_i \mu_{ij}^D \delta w_j = \frac{1}{2} n_i \mu_{ij}^D \varepsilon_{jkl} \left(n_k D\delta u_l + D_k \delta u_l \right) \tag{A3.8}$$

$$n_i \mu\delta_{ij} \delta w_j = \frac{1}{2} n_i \,(\mu\delta_{ij}) \varepsilon_{jkl} \left(n_k D\delta u_l + D_k \delta u_l \right) \tag{A3.9}$$

We remark that

$$\begin{aligned}\frac{1}{2} n_i \mu_{ij}^D \varepsilon_{jkl} D_k \delta u_l &= \frac{1}{2} D_k \left(n_i \mu_{ij}^D \varepsilon_{jkl} \delta u_l \right) - \frac{1}{2} D_k (n_i) \mu_{ij}^D \varepsilon_{jkl} \delta u_l - \frac{1}{2} n_i D_k (\mu_{ij}^D) \varepsilon_{jkl} \delta u_l = \\ &= \frac{1}{2} D_k \Phi_k - \frac{1}{2} D_k (n_i) \left(\frac{2}{3} \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right) \delta u_l - \frac{1}{2} n_i D_k \left(\frac{2}{3} \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right) \delta u_l\end{aligned} \tag{A3.10}$$

with

$$\Phi_k = n_i \mu_{ij}^D \varepsilon_{jkl} \delta u_l \tag{A3.11}$$

for which the following identity holds:

$$D_k \Phi_k = D_p n_p n_k \Phi_k - n_q \varepsilon_{qpm} \partial_p \left(\varepsilon_{mlk} n_l \Phi_k \right) \tag{A3.12}$$

With

$$A_m = \varepsilon_{mlk} n_l \Phi_k \tag{A3.13}$$

we get that over a closed surface

$$\oint_S n_q \varepsilon_{qpm} \partial_p A_m dS = 0 \tag{A3.14}$$

Using the above identity and Eq. (A3.10), the integral of Eq. (A3.8) results to the following expression:

$$\begin{aligned}W_\mu &= \int_{\partial V} n_i \mu_{ij}^D \delta w_j = \int_{\partial V} \left\{ \frac{1}{2} n_i n_k \left(\frac{2}{3} \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right) \right\} D\delta u_l dS + \\ &+ \int_{\partial V} \left(D_p n_p n_k \Phi_k - \frac{1}{2} D_k (n_i) \left(\frac{2}{3} \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right) - \frac{1}{2} n_i D_k \left(\frac{2}{3} \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right) \right) \delta u_l dS\end{aligned}$$

or

$$W_\mu = \int_{\partial V} \left\{ \frac{1}{3} n_i\, n_k \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right\} D\delta u_l dS + $$
$$+ \int_{\partial V} \left\{ \frac{1}{3} D_p n_p n_k n_i \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) - \frac{1}{3} D_k (n_i) \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) - \frac{1}{3} n_i\, D_k \left(\hat{\mu}_{kli} - \hat{\mu}_{lki} \right) \right\} \delta u_l dS \tag{A3.15}$$

The integral of Eq. (A3.9) becomes:

$$\int_{\partial V} n_i\, \delta_{ij} \mu \delta w_j = \int_{\partial V} \left\{ \frac{1}{2} n_i\, (\mu \delta_{ij}) \varepsilon_{jkl} \left(n_k D\delta u_l + D_k \delta u_l \right) \right\} dS =$$
$$= \int_{\partial V} \left(\frac{1}{2} n_i\, (\mu \delta_{ij}) \varepsilon_{jkl} n_k \right) D\delta u_l dS +$$
$$+ \int_{\partial V} \left\{ \frac{1}{2} D_k \left(n_i\, (\mu \delta_{ij}) \varepsilon_{jkl} \delta u_l \right) - \frac{1}{2} D_k (n_i) (\mu \delta_{ij}) \varepsilon_{jkl} \delta u_l - \frac{1}{2} n_i\, D_k \left((\mu \delta_{ij}) \varepsilon_{jkl} \right) \delta u_l \right\} dS \tag{A3.16}$$
$$= \int_{\partial V} \left\{ \frac{1}{2} D_p n_p n_k n_i\, (\mu \delta_{ij}) \varepsilon_{jkl} \delta u_l - \frac{1}{2} D_k (n_i) (\mu \delta_{ij}) \varepsilon_{jkl} \delta u_l - \frac{1}{2} n_i\, D_k \left((\mu \delta_{ij}) \varepsilon_{jkl} \right) \delta u_l \right\} dS$$
$$= \int_{\partial V} \left\{ -\frac{1}{2} D_k (n_i) (\mu \delta_{ij}) \varepsilon_{jkl} - \frac{1}{2} n_i\, D_k \left((\mu \delta_{ij}) \varepsilon_{jkl} \right) \right\} \delta u_l dS$$

And therefore the integral of the second term of Eq.(A3.1) yields

$$\boxed{\begin{aligned} &\int_{\partial V} n_i \mu_{ij} \delta w_j dS = \int_{\partial V} \left\{ \frac{1}{3} n_i\, n_j \left(\hat{\mu}_{jki} - \hat{\mu}_{kji} \right) \right\} D\delta u_k dS + \\ &+ \int_{\partial V} \left\{ \frac{1}{3} D_p n_p n_i n_j \left(\hat{\mu}_{jki} - \hat{\mu}_{kji} \right) - \frac{1}{3} D_j (n_i) \left(\hat{\mu}_{jki} - \hat{\mu}_{kji} \right) - \frac{1}{3} n_i\, D_j \left(\hat{\mu}_{jki} - \hat{\mu}_{kji} \right) \right\} \delta u_k dS - \\ &- \int_{\partial V} \left\{ \frac{1}{2} D_k (n_i) (\mu \delta_{ij}) \varepsilon_{jkl} + \frac{1}{2} n_i\, D_k \left((\mu \delta_{ij}) \varepsilon_{jkl} \right) \right\} \delta u_l dS \end{aligned}} \tag{A3.17}$$

- The **third term** of the left hand side of Eq. (A3.1) is analyzed as follows:

$$\begin{aligned} n_i \mu_{ijk} \delta \varepsilon_{jk} &= n_i \mu_{ijk} \partial_j \delta u_k = n_i \mu_{ijk} n_j D\delta u_k + n_i \mu_{ijk} D_j \delta u_k = \\ &= n_i \mu_{ijk} n_j D\delta u_k + D_j (n_i \mu_{ijk} \delta u_k) - D_j n_i \mu_{ijk} \delta u_k - n_i D_j \mu_{ijk} \delta u_k \end{aligned} \tag{A3.18}$$

and its integral becomes

$$\boxed{\begin{aligned}\int_{\partial V} n_i\mu_{ijk}\delta\varepsilon_{jk}dS = &\int_{\partial V}\frac{1}{3}n_i\left(\hat{\mu}_{ijk}+\hat{\mu}_{jki}+\hat{\mu}_{kij}\right)n_j D\delta u_k dS+\\&+\int_{\partial V}\left\{\frac{1}{3}D_p n_p n_i\left(\hat{\mu}_{ijk}+\hat{\mu}_{jki}+\hat{\mu}_{kij}\right)n_j\right\}\delta u_k dS-\\&-\int_{\partial V}\left\{\frac{1}{3}D_j n_i\left(\hat{\mu}_{ijk}+\hat{\mu}_{jki}+\hat{\mu}_{kij}\right)+\frac{1}{3}n_i D_j\left(\hat{\mu}_{ijk}+\hat{\mu}_{jki}+\hat{\mu}_{kij}\right)\right\}\delta u_k dS\end{aligned}} \tag{A3.19}$$

Finally by substituting Eqs. (A3.5), (A3.17) and (A3.19) into the left hand side of Eq. (A3.1) we get

$$\begin{aligned}&\int_{\partial V}\left(\tau_{ij}\delta u_j+\mu_{ij}\delta w_j+\mu_{ijk}\delta\varepsilon_{jk}\right)n_i\,dS\\&=\int_{\partial V}\frac{1}{3}n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)n_j D\delta u_k dS+\int_{\partial V}\left\{n_j\hat{\tau}_{ij}-\frac{1}{3}n_j\partial_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)+\frac{1}{3}D_p n_p n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)n_j-\right.\\&\left.-\frac{1}{3}D_j n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)+\frac{1}{3}n_i D_j\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)\right\}\delta u_k dS-\\&-\int_{\partial V}\left\{\frac{1}{2}(\mu\delta_{ij})\varepsilon_{jlk}D_l n_i+\frac{1}{2}n_i D_l(\mu\delta_{ij})\varepsilon_{jlk}+\frac{1}{2}n_j\varepsilon_{ijk}\delta_{li}\partial_l\mu\right\}\delta u_k dS\end{aligned} \tag{A3.20}$$

and

$$\underline{\frac{1}{2}n_i D_l(\mu\delta_{ij})\varepsilon_{jlk}+\frac{1}{2}n_j\varepsilon_{ijk}\delta_{li}\partial_l\mu=\frac{1}{2}n_j(D_l-\partial_l\mu)\varepsilon_{jlk}=\frac{1}{2}n_j n_l\varepsilon_{jlk}D\mu=0} \tag{A3.21}$$

And therefore

$$\boxed{\begin{aligned}&\int_{\partial V}\left(\tau_{ij}\delta u_j+\mu_{ij}\delta w_j+\mu_{ijk}\delta\varepsilon_{jk}\right)n_i\,dS=\\&\int_{\partial V}\frac{1}{3}n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)n_j D\delta u_k dS+\\&\int_{\partial V}\left\{n_j\hat{\tau}_{ij}-\frac{1}{3}n_j\partial_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)+\frac{1}{3}D_p n_p n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)n_j-\right.\\&\quad\left.-\frac{1}{3}D_j n_i\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)+\frac{1}{3}n_i D_j\left(\hat{\mu}_{ijk}+2\hat{\mu}_{jki}\right)\right\}\delta u_k dS-\\&-\int_{\partial V}\left\{\frac{1}{2}(\mu\delta_{ij})e_{jlk}D_l n_i\right\}\delta u_k dS\end{aligned}} \tag{A3.22}$$

And by substituting Eq. (A3.22) into Eq.(A3.1) we get Eq.(52).

We can also formulate the relations between P_k and Q_k in terms of the "true" forces:

$$P_k = t_k + \frac{1}{2} e_{jlk} \left(D_p n_p n_l - D_l \right) m_j + \left(D_p n_p n_j - D_j \right) R_{jk} \tag{A.3.23}$$

$$Q_k = \frac{1}{2} e_{jlk} n_l m_j + n_j R_{jk} \tag{A3.24}$$

Appendix IV: On a field and its normal derivative along a curve

Let us consider for simplicity a two-dimensional situation and let in the plane (x, y) a curve (Γ), which is described by its analytic function

$$(\Gamma): \quad y = f(x) \tag{A4.1}$$

This curve (Γ), is that part of the boundary of a body (**B**) where some or all displacement components are given as boundary conditions, say

$$u_\alpha(x, f(x)) = \bar{u}_\alpha(x) \tag{A4.2}$$

Let us consider now two neighbouring points A and B on that curve (Γ) with the co-ordinates

$$A(x_A, y_A) \quad , \quad B(x_B = x_A + \Delta x, y_B = y_A + \Delta y) \tag{A4.3}$$

where

$$\frac{\Delta y}{\Delta x} \approx \left(\frac{df}{dx} \right)_A \tag{A4.4}$$

For the computation of the values of the function $u_\alpha(x, y)$ at the interior point $C(x_B, y_A) \in$ **B,** we expand this function in Taylor series around the points A and B and we keep only the two first terms (Fig. A4.1). We remark that we can compute the value at point C by either going from A directly to C on the horizontal line $y = y_A$ or indirectly by first going to point B along the curve (Γ) and then moving vertically along the line $x = x_B$ from B to C; i.e.:

$$u_\alpha(C) \approx u_\alpha(A) + \left(\frac{\partial u_\alpha}{\partial x} \right)_A \Delta x \tag{A4.5}$$

Since

$$\left(\frac{\partial u_\alpha}{\partial y}\right)_B = \left(\frac{\partial u_\alpha}{\partial y}\right)_A + \left(\frac{\partial}{\partial x}\left(\frac{\partial u_\alpha}{\partial x}\right)_A + \frac{\partial}{\partial y}\left(\frac{\partial u_\alpha}{\partial y}\right)_A \frac{df}{dx}\right)\Delta x \tag{A4.6}$$

we get from Eq. (A4.5) that

$$\begin{aligned} u_\alpha(C) &\approx u_\alpha(B) - \left(\frac{\partial u_\alpha}{\partial y}\right)_B \Delta y \\ &\approx u_\alpha(A) + \left(\frac{\partial u_\alpha}{\partial x}\right)_A \Delta x + \left(\frac{\partial u_\alpha}{\partial y}\right)_A \Delta y - \left(\frac{\partial u_\alpha}{\partial y}\right)_B \Delta y \\ &= u_\alpha(A) + \left(\frac{\partial u_\alpha}{\partial x}\right)_A \Delta x + O\left(\Delta x^2\right) \end{aligned} \tag{A4.7}$$

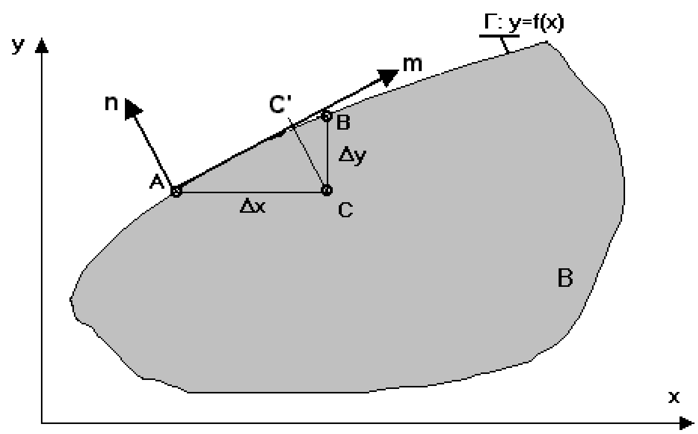

Figure A4.1

Thus the value of the function $u_\alpha(x, y)$ at point C does not depend on the path,

$$u_\alpha(C) \approx u_\alpha(A) + \left(\frac{\partial u_\alpha}{\partial x}\right)_A \Delta x \approx u_\alpha(B) - \left(\frac{\partial u_\alpha}{\partial y}\right)_B \Delta y \tag{A4.8}$$

Moreover we observe that the value of $\left(\frac{\partial u_\alpha}{\partial y}\right)_A$ and of $\left(\frac{\partial u_\alpha}{\partial x}\right)_A$ are interrelated.

Indeed from

$$u_\alpha(B) \approx u_\alpha(A) + \left(\frac{\partial u_\alpha}{\partial x}\right)_A \Delta x + \left(\frac{\partial u_\alpha}{\partial y}\right)_A \Delta y \tag{A4.9}$$

we get

$$\left(\frac{d\bar{u}_\alpha}{dx}\right)_A \approx \frac{u_\alpha(B) - u_\alpha(A)}{\Delta x} \approx \left(\frac{\partial u_\alpha}{\partial x}\right)_A + \left(\frac{\partial u_\alpha}{\partial y}\right)_A \left(\frac{df}{dx}\right)_A \tag{A4.10}$$

Thus for the continuation of the considered function $u_\alpha(x,y)$ from a boundary point $A \in (\Gamma)$ to a point C, in the interior of body **B,** we need:

1. The data for the function $u_\alpha(x,y)$ on the boundary, Eq. (A4.2).
2. The information concerning its gradient in only one direction.

In particular we may chose point B to coincide with the normal projection of point C on the curve, $B \equiv C'$. From the figure we can see immediately that,

$$u_\alpha(C) \approx u_\alpha(A) + \left(\frac{d\bar{u}_\alpha}{dx}\right)_A \Delta m + \left(\frac{\partial u_\alpha}{\partial n}\right)_{C'} \Delta n \approx u_\alpha(C') + \left(\frac{\partial u_\alpha}{\partial n}\right)_{C'} \Delta n \tag{A4.11}$$

where m and n are the directions tangential and normal to the boundary in the vicinity of the considered points. This means that for the determination of the function $u_\alpha(x,y)$ in the interior domain, starting from boundary data we must:

a) specify the function along the boundary curve $u_\alpha(x, f(x)) = \bar{u}_\alpha(x)$, which will allow us to compute

$$u_\alpha(C') = u_\alpha(A) + \left(\frac{d\bar{u}_\alpha}{dx}\right)_A \Delta m \tag{A4.12}$$

b) we must know the value of the normal derivative of the considered function along the boundary

$$Du_\alpha(A) = \left(\frac{\partial u_\alpha}{\partial n}\right)_A \tag{A4.13}$$

so that

$$u_\alpha(C) \approx u_\alpha(A) + \left(\frac{d\bar{u}_\alpha}{dx}\right)_A \Delta m + Du_\alpha \Delta n \qquad \text{(A4.15)}$$