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an EU SOCRATES short course
on

Engineering Continuum Mechanics: traffic flow and shallow water waves

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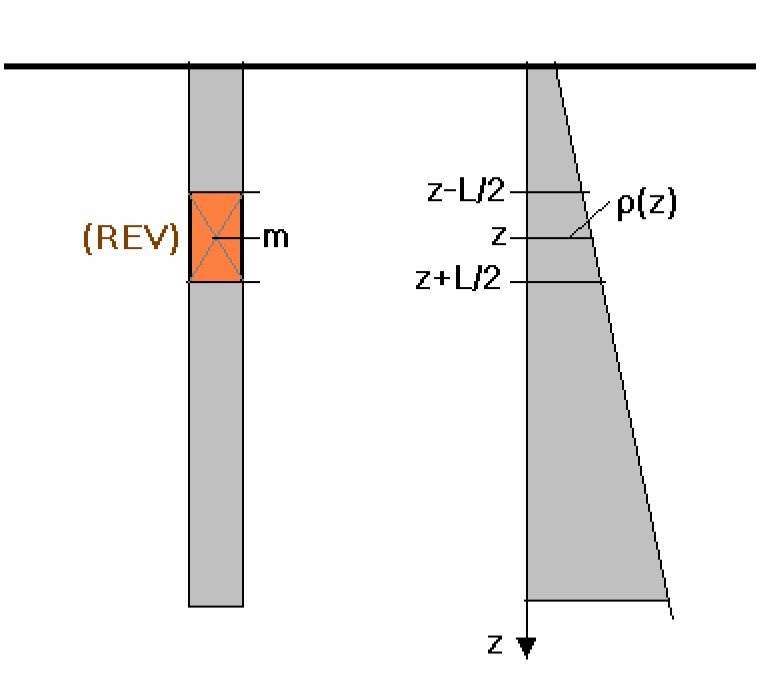
(http://geolab.mechan.ntua.gr)

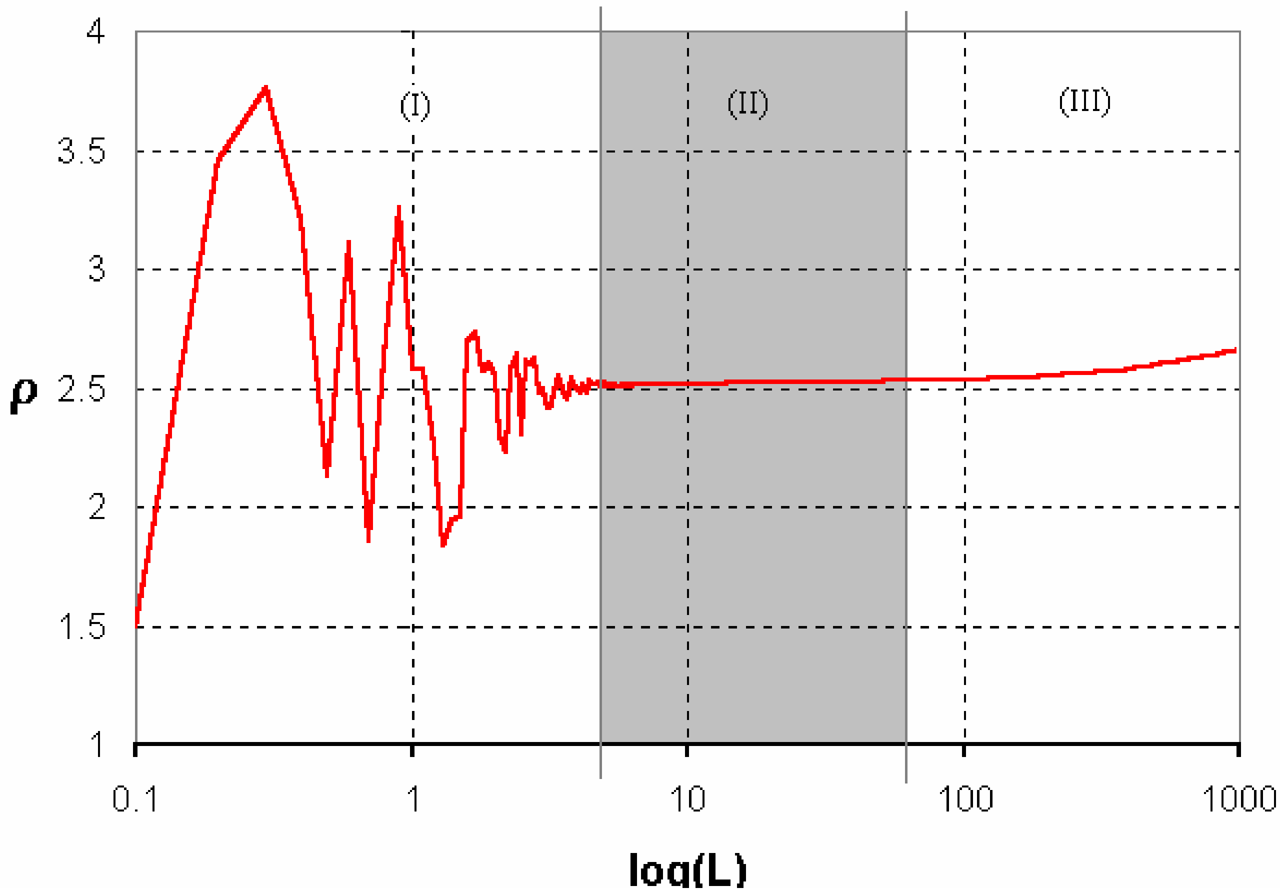
- **natural phenomena**
- **mathematical modeling-the continuum assumption**
- **the elementary traffic-flow theory**
- **St. Venant's "shallow-water" theory**
- **"shallow-water" model of granular flows**

the continuum assumption



$$< \rho >_{z,L} = \rho(z)$$

Leonhard Euler 1707-1783



L. Prandtl and O.G. Tietjens, *Fundamentals of Hydro- and Aerodynamics*, Dover , 1934.

The **Lagrangian Approach** to continuum mechanics emphasizes the particulate (material) description of a motion

The pathlines of car-particles flowing in a highway: Red paths belong to receding particles; white paths to approaching particles. In Fluid Mechanics these lines are called the pathlines. We notice however this snapshot corresponds to a photograph with “long exposure”. Thus “long” exposures yield the Lagrangian view, whereas “short” exposures the Eulerian view

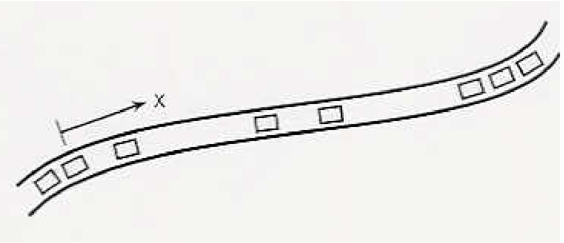
The **Eulerian Approach** to continuum mechanics emphasizes the spatial description of a motion

Ground-wind velocities meteorologic map of a given place at a given instant. In Fluid Mechanics the integral curves of this velocity field are called the streamlines

$$v = v(x, y, t)$$

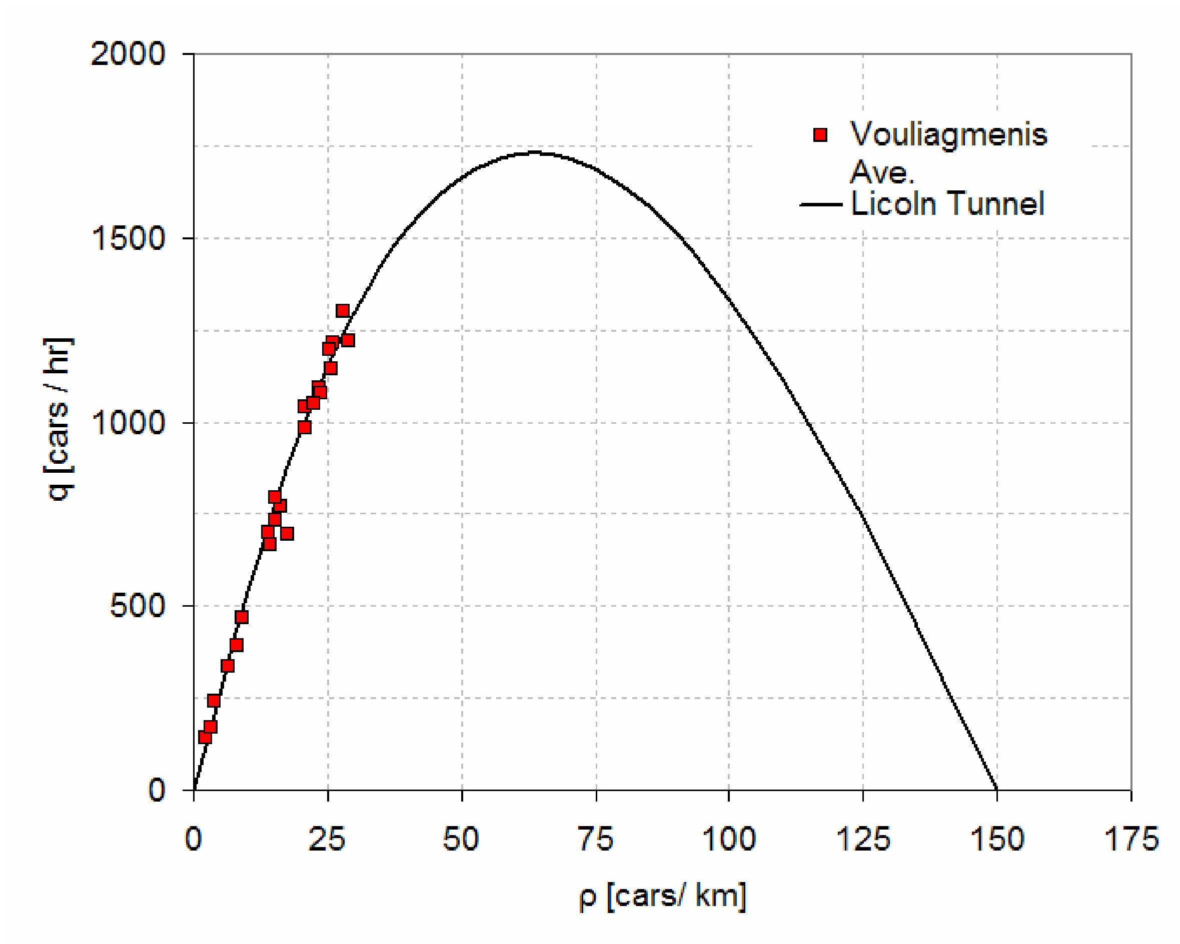
The Elementary Traffic-Flow Theory (eulerian)

(M.J. Lighthill and G.B. Whitham 1955)



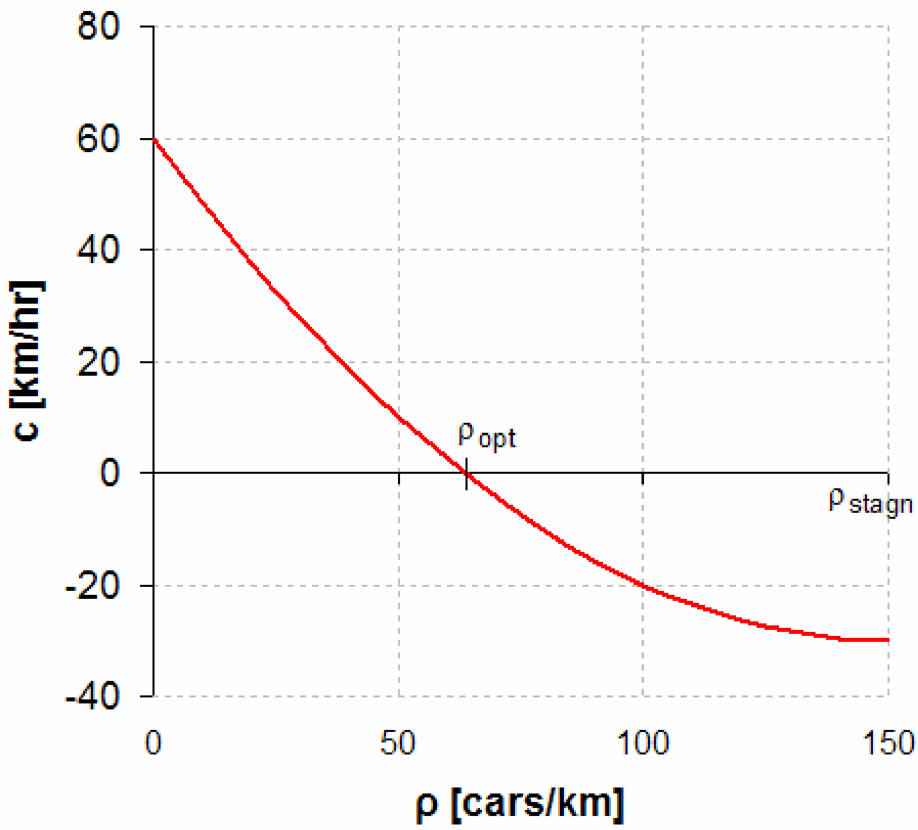
- Linear density of vehicles: $\rho = \rho(x, t)$, $[\rho] = \text{cars / km}$
- (mean) speed of cars: $v = v(x,t)$, $[v] = \text{km / hr}$
- "mass" balance: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$
- need for a "closure"; eg. for the car flow-rate: $q = \rho v = Q(\rho, \ldots)$

empirical constitutive law



the celerity

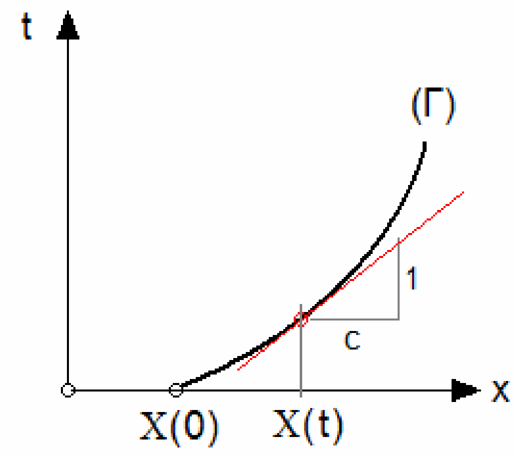
- $v = V(\rho)$
- $\boxed{q = Q(\rho) = \rho\, V(\rho)}$; e.g. $q = Q(\rho) = 60\,\rho - \frac{3}{5}\rho^2 + \frac{1}{750}\rho^3$
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} Q(\rho) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{dQ}{d\rho}\frac{\partial \rho}{\partial x} = 0$
- $\boxed{\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0}$; $c = C(\rho) = \frac{dQ}{d\rho}$



the traffic flow problem

Given: $\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \quad (*)$

- Assume that at time $t = 0$ the *initial condition*: $\rho(x,0) = r(x)$.
- Compute the space-time evolution of $\rho(x,t)$



Solution: Consider in the **plane of events** $O(x,t)$ a curve

$$(\Gamma): \quad x = X(t) \ \text{ s.t. } \frac{dX}{dt} = c\left(H\left(X(t),t\right)\right)$$

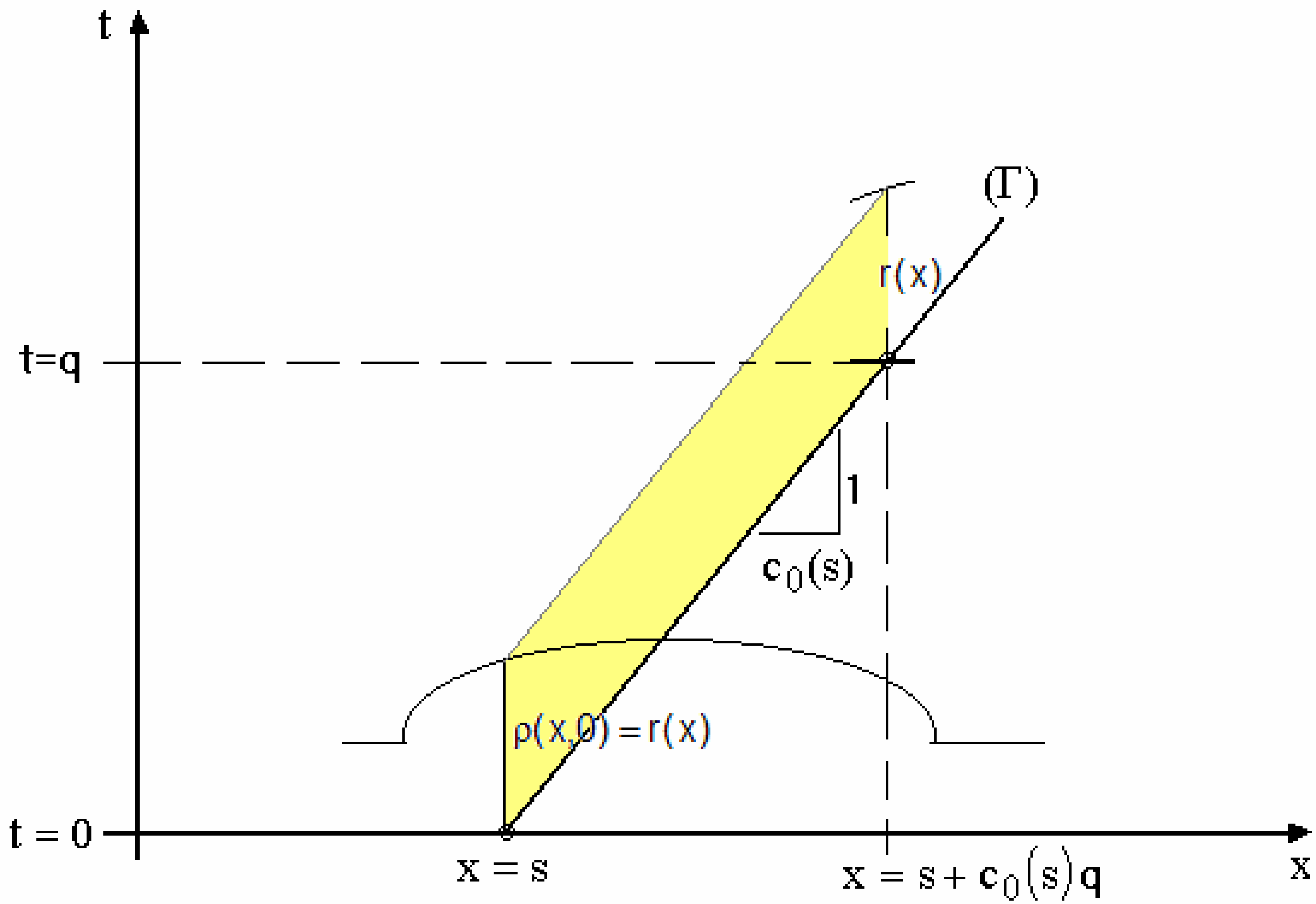
Along (Γ) the density is a function only of time t: $\rho = \rho\left(X(t), t\right) = \hat{\rho}(t)$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dX}{dt} = 0 \quad \Rightarrow \quad \rho = \text{const}$$

$$\Rightarrow c(\hat{H}) = \text{const.} \quad \Rightarrow \quad \frac{dX}{dt} = \text{const.}$$

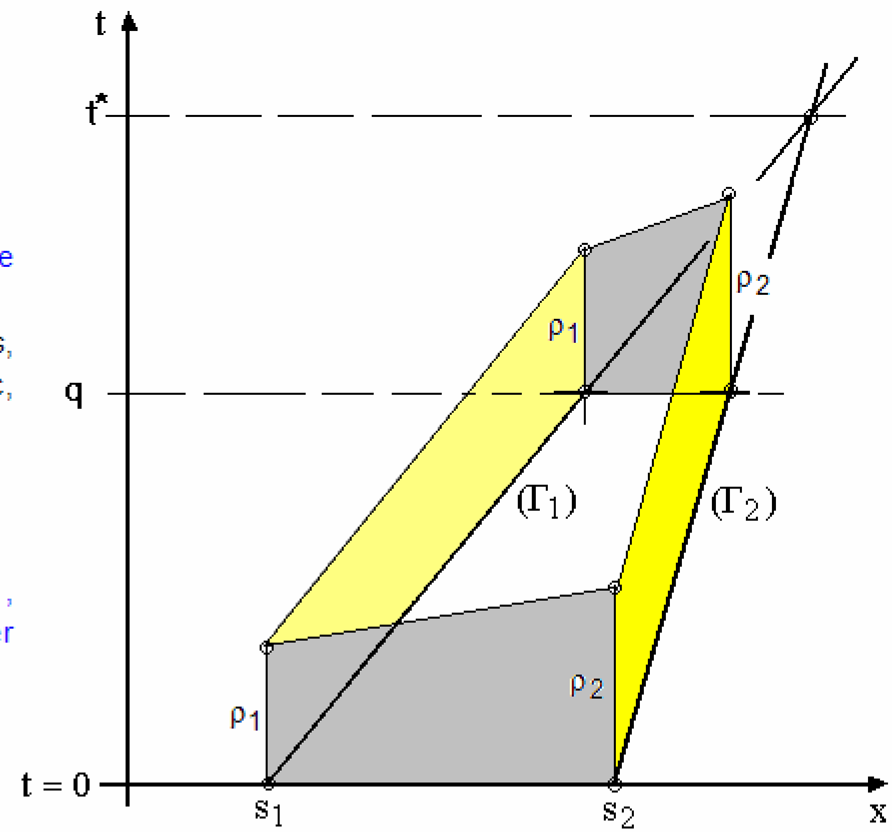
(Γ) is a **straight line** in the plane $O(x,t)$ of events.

(Γ) is called a **characteristic line** or simply a **characteristic** of equation (*)[1].

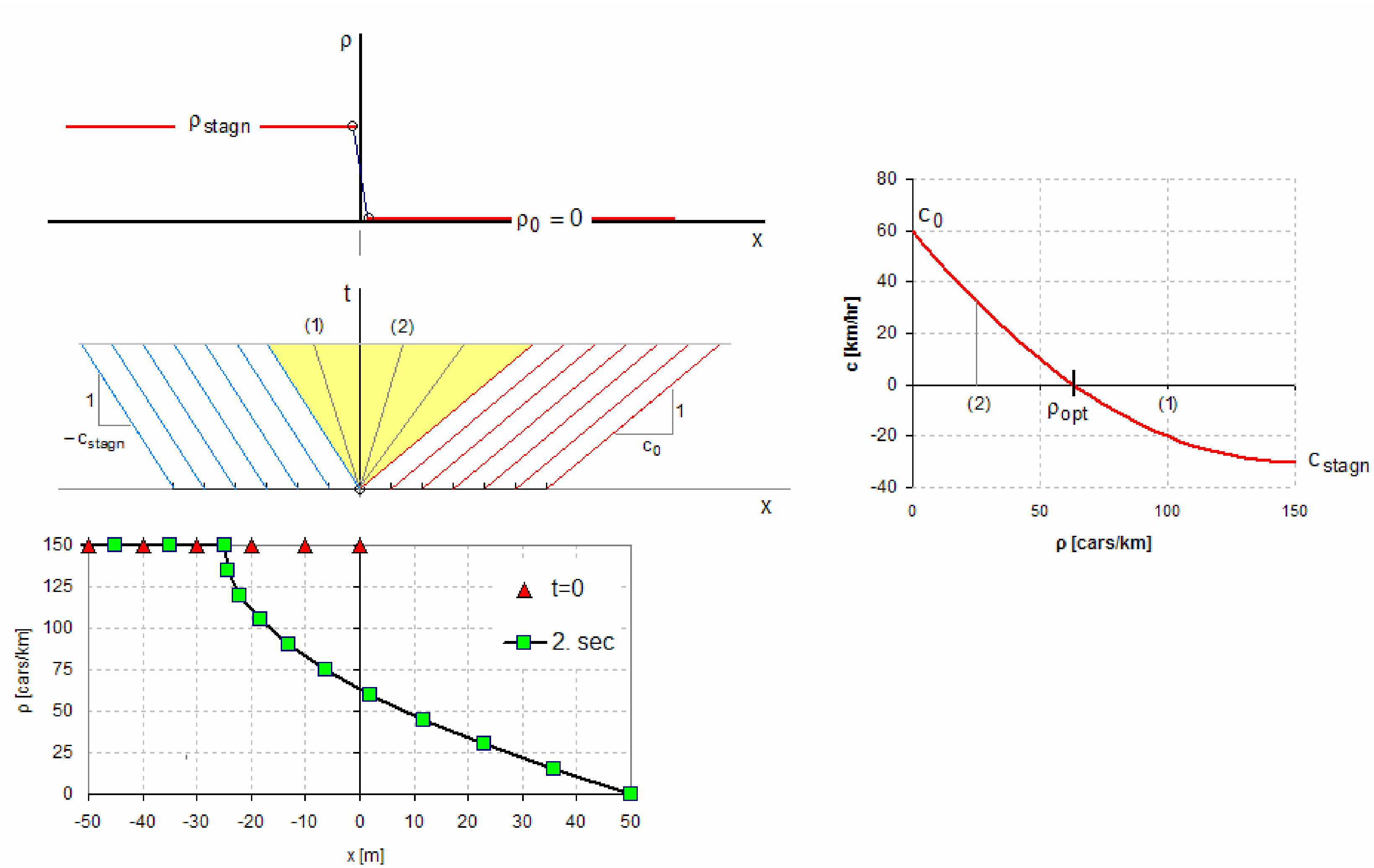

t
t=q
t = 0
(Γ)
r(x)
1
c0(s)
ρ(x,0) = r(x)
x = s
x = s + c0(s)q
x

'Method of Characteristics':

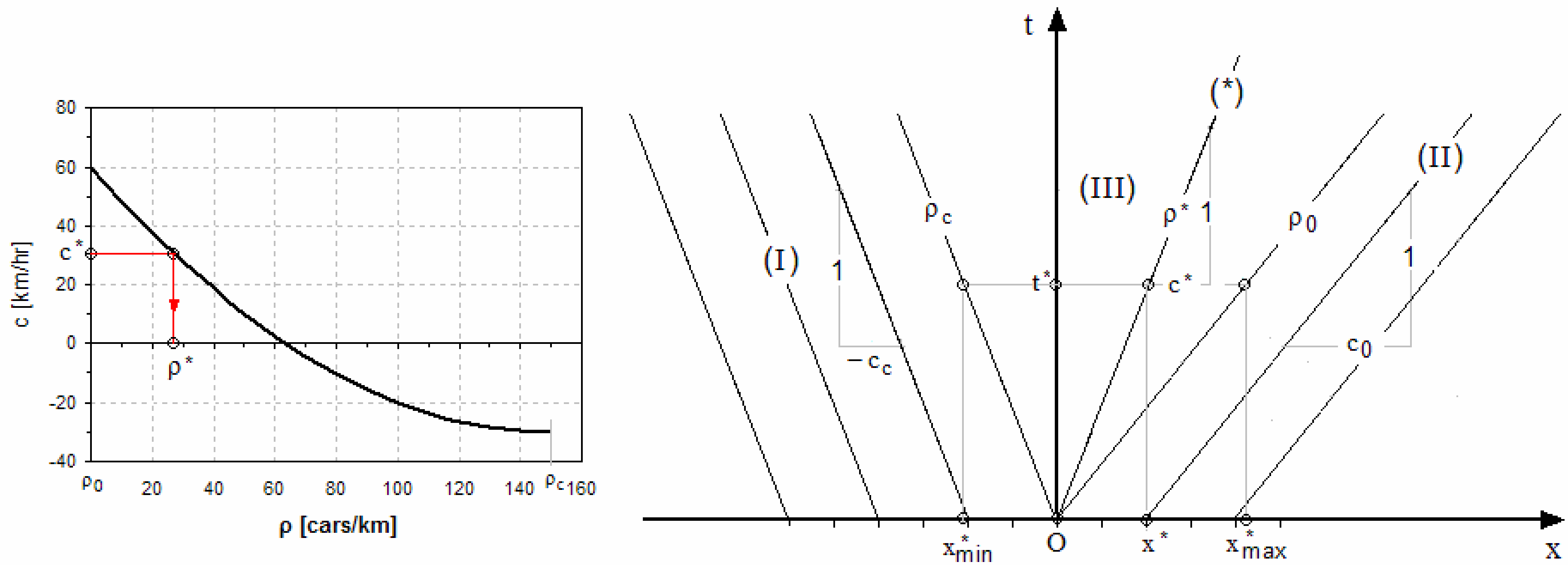
1. The x-axis is discretised.
2. We evaluate the initial condition for the height at the discrete points on the x-axis, $\rho(x,0) = r(x)$
3. At any individual point $x = s$, along the x-axis ($t = 0$), we compute the propagation velocity, $c_0(s) = c\big(r(s)\big)$
4. In the plane of events $O(x,t)$ and through the point on the x-axis, with co-ordinates $(s,0)$ we draw the straight-line characteristic, $(\Gamma):\quad x = s + c_0(s)t$
5. Along this characteristic straight line the information concerning the density is transferred constant, $(\Gamma):\quad \rho(s + c_0(s)t, t) = \rho(s,0) = r(s)$
6. If we want to evaluate the density profile at a given time $t = q > 0$, we intersect the characteristics with the line, $t = q$, and plot over the intersection points the values for ρ which they carry.



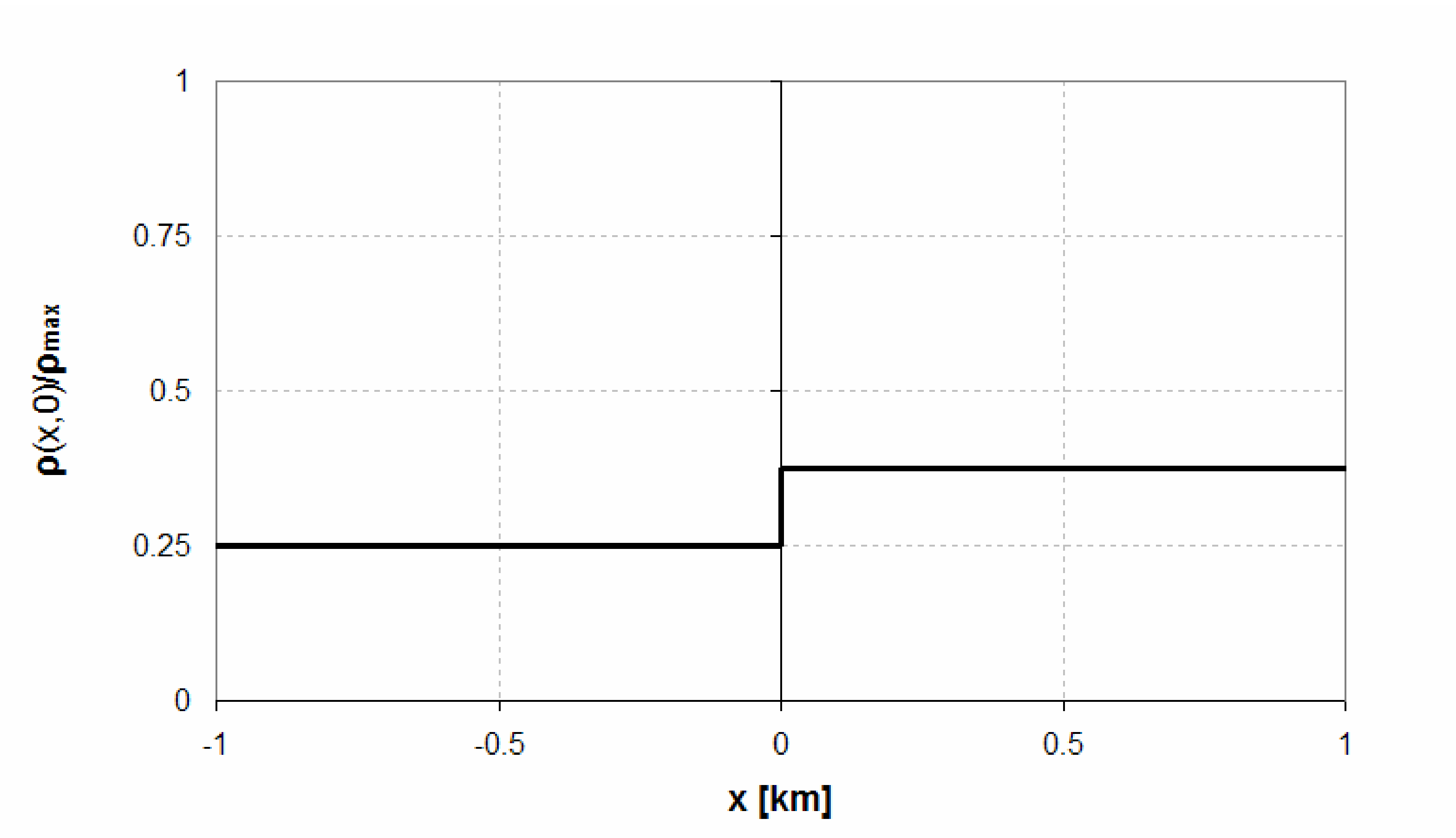
The traffic light problem

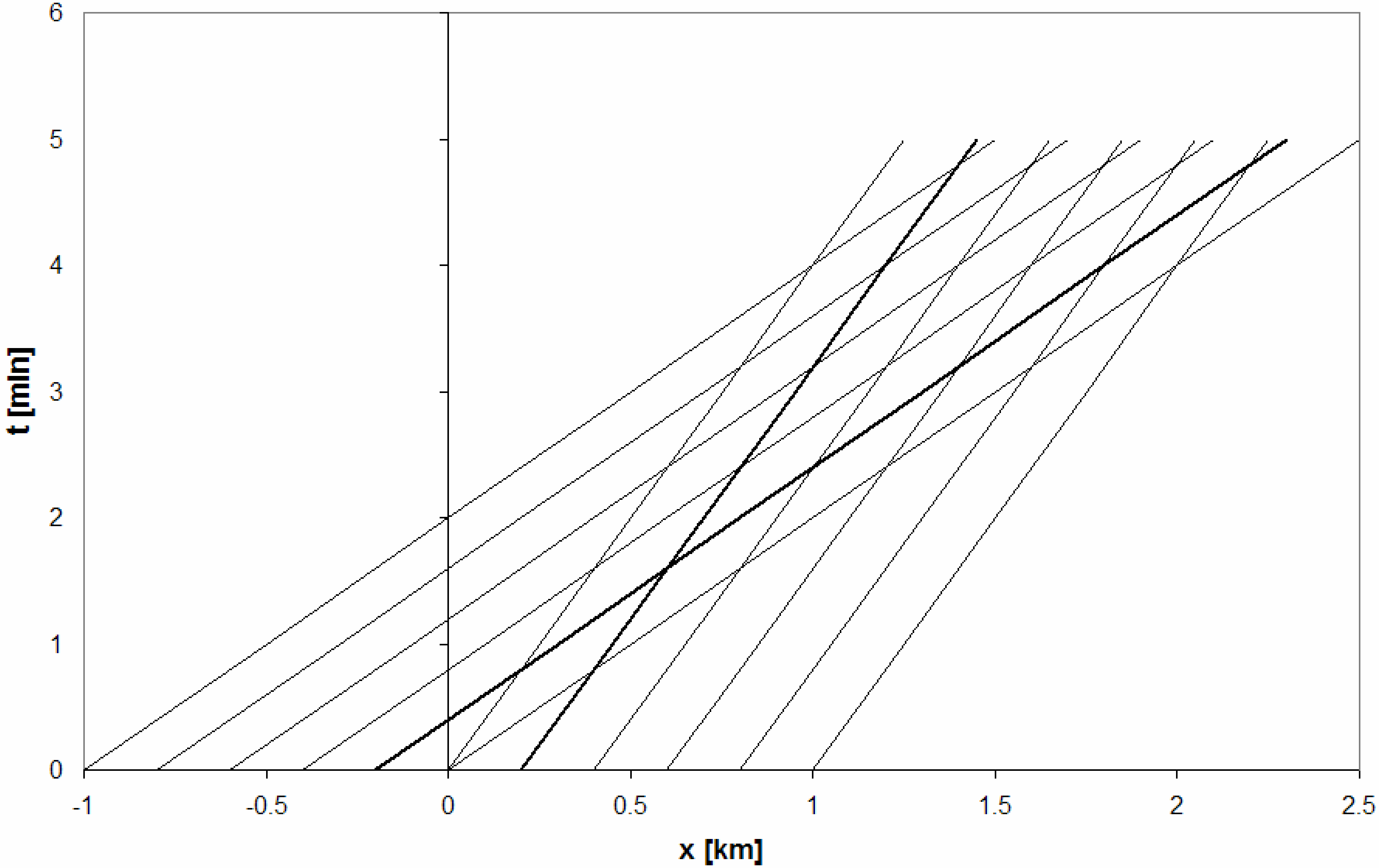


the expansion fan

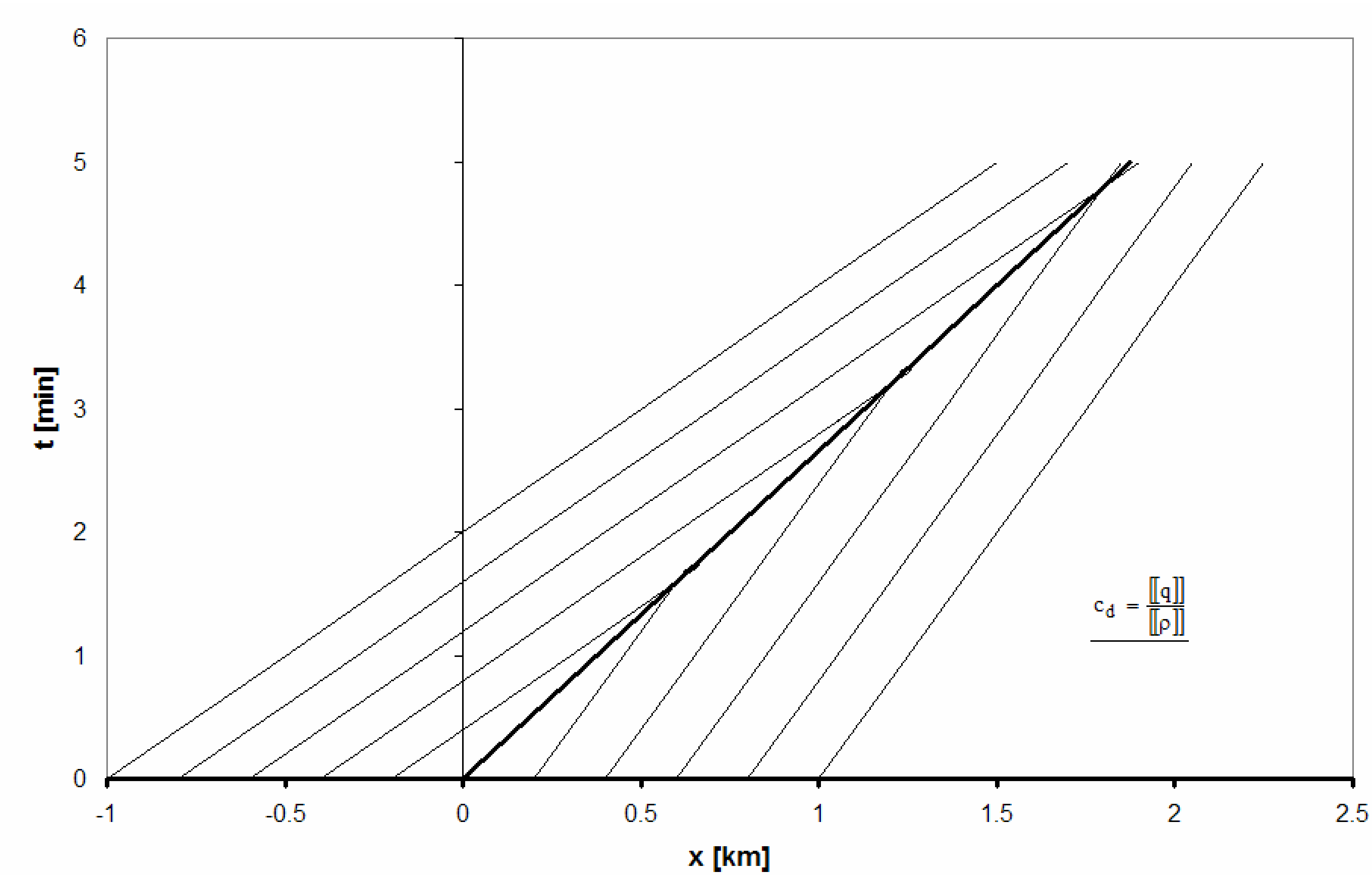


the shock wave





t [min]
x [km]
6
5
4
3
2
1
0
-1
-0.5
0
0.5
1
1.5
2
2.5



6
5
4
3
2
1
0
t [min]
-1
-0.5
0
0.5
1
1.5
2
2.5
x [km]
$c_d = \frac{[[q]]}{[[\rho]]}$

1st Modification: “Viscosity” correction (the driver looks ahead; stabilizing)

2nd Modification: Reaction time (destabilizing)

Viscosity correction

If the flow ahead is getting denser or looser, the driver is able to adjust the speed of the vehicle accordingly:

$$v = V(\rho) - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x}$$

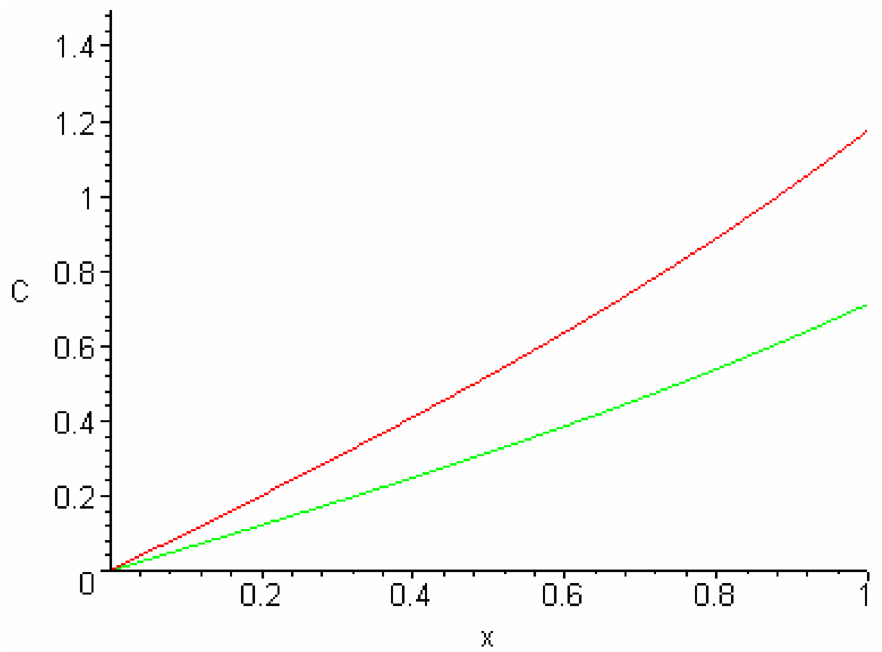
$$\nu = \ell_c v_0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)\frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \quad \text{(BURGER Eq.)}$$

Canonical form of the Burger equation

$$C = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) \Rightarrow$$

$$\frac{\partial C}{\partial t} + C\frac{\partial C}{\partial x} = \nu\frac{\partial^2 C}{\partial x^2}$$



Reaction time

$$v = V\left(\rho(x,t-\tau)\right) \approx V(\rho) - \tau V'(\rho)\frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = \tau \frac{\partial}{\partial x}\left(\rho V'(\rho)\frac{\partial \rho}{\partial t}\right)$$

$$\rho = \rho^* + \tilde{\rho}(x,t)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + c^* \frac{\partial \tilde{\rho}}{\partial x} = \tau \rho^* V'(\rho^*)\frac{\partial^2 \tilde{\rho}}{\partial x \partial t}$$

Linear stability analysis

$$\tilde{\rho} = \mathrm{Re}\left(\exp(ikx + st)\right) \quad , \quad i = \sqrt{-1}$$

$$s = \frac{\tau k^2 \rho^* c V'(\rho^*) - ikc^*}{1 + \left(\tau k \rho^* V'(\rho^*)\right)^2}$$

with V’<0 instability is predicted at heavy traffic conditions (c<0)

St. Venant's Shallow-Water Theory (eulerian)

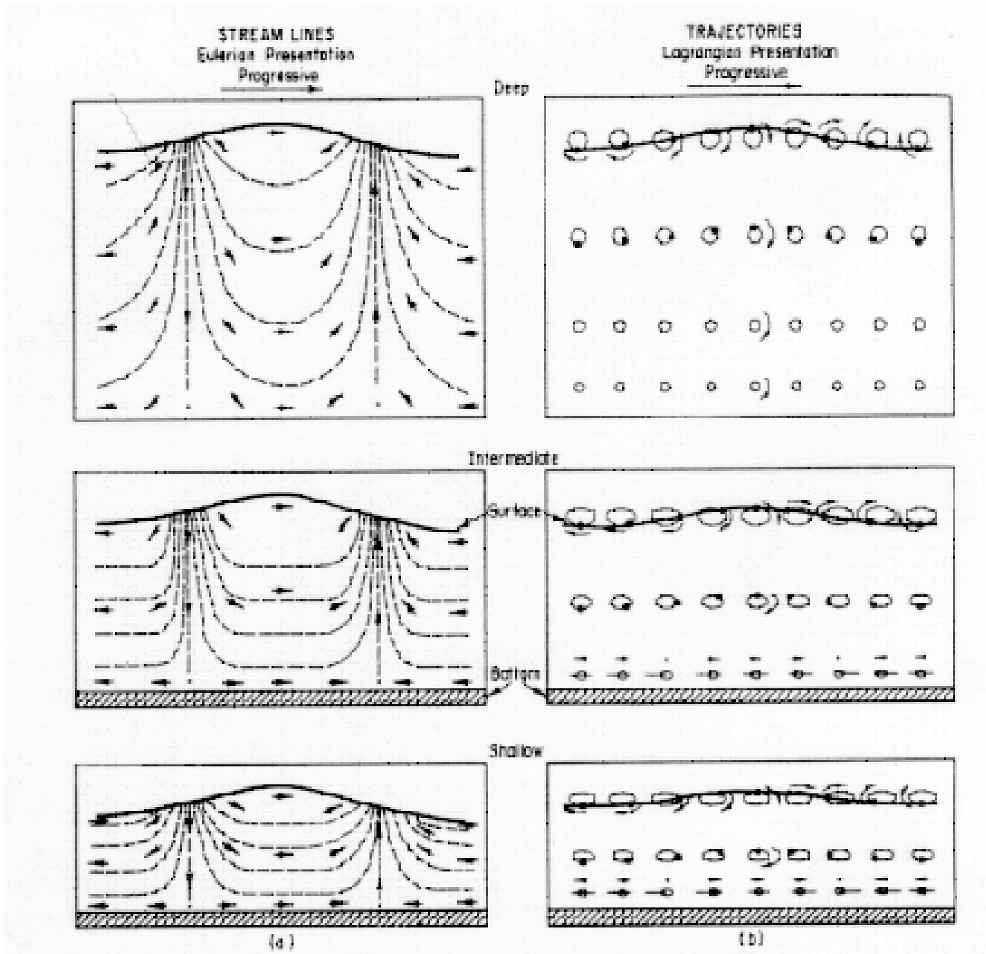
De Barr Saint-Venant, A.J.C. (1850). Mémoire sur des formules nouvelles pour la solution des problèmes ralitfs aux eaux courantes. C.R. Acad. Sc., Paris, 31, 283.

De Barr Saint-Venant, A.J.C. (1871). Théorie du movement non-permanent des eaux avec applications aux crues des rivieres et a l' introduction des marées dans leurs lit. C.R. Acad. Sc., Paris, 73, 147-154.

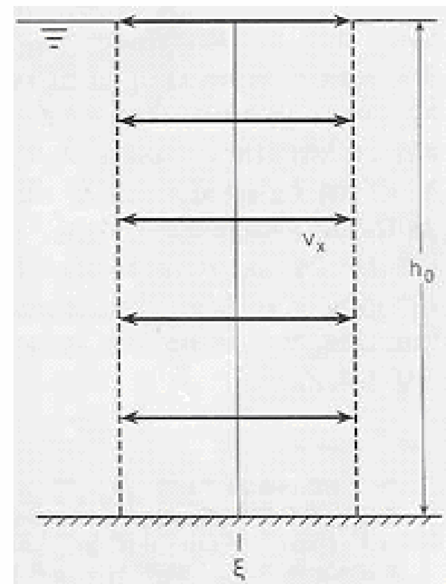
The "shallow water theory" is traced originally to St. Venant in the mid 19th century and its successful application to open channel flow and to river dynamics is extensively presented in standard textbooks.

Within the shallow water theory the governing mass and momentum balance equations of continuum mechanics are averaged over space and time in an appropriate manner so that simplified and mathematically more tractable differential equations are derived. Space averaging is done over the height of the flowing mass, whereas time averaging is done primarily in order to account for the effect of fluctuations at the interface between the flowing mass and its stationary base.

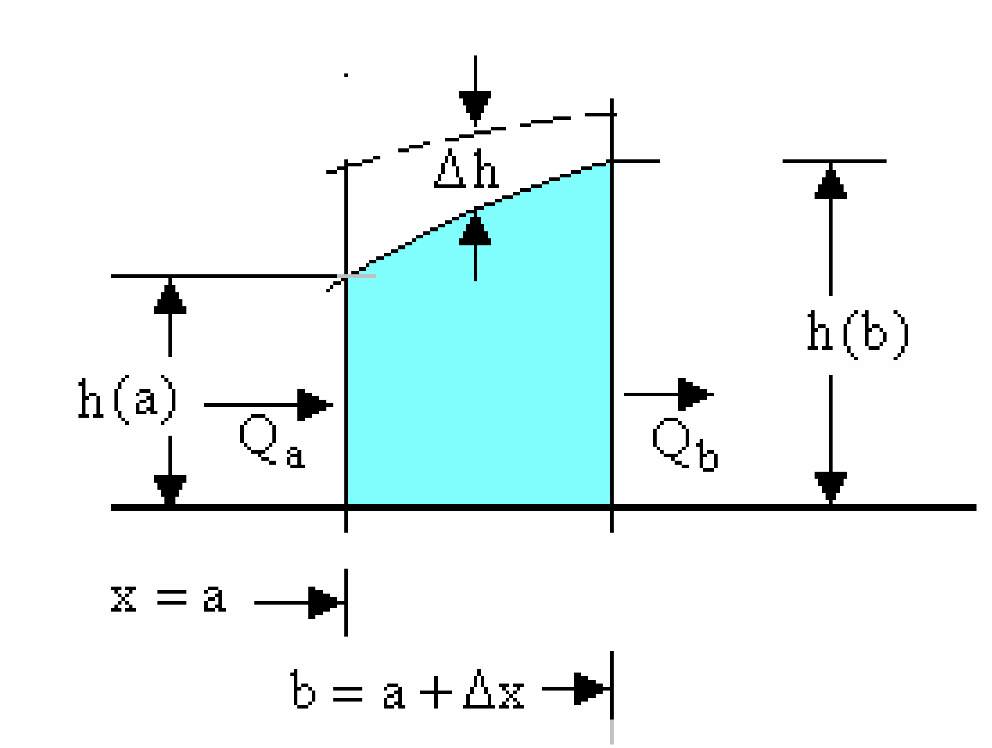
Shallow-water limit



shallow water

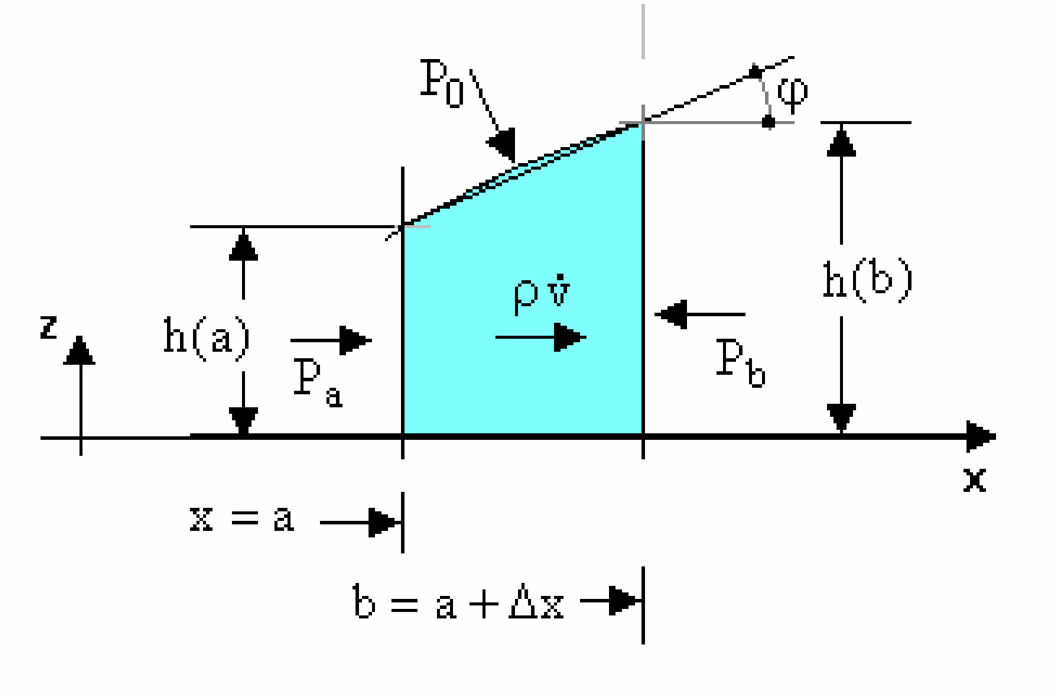


Storage Equation



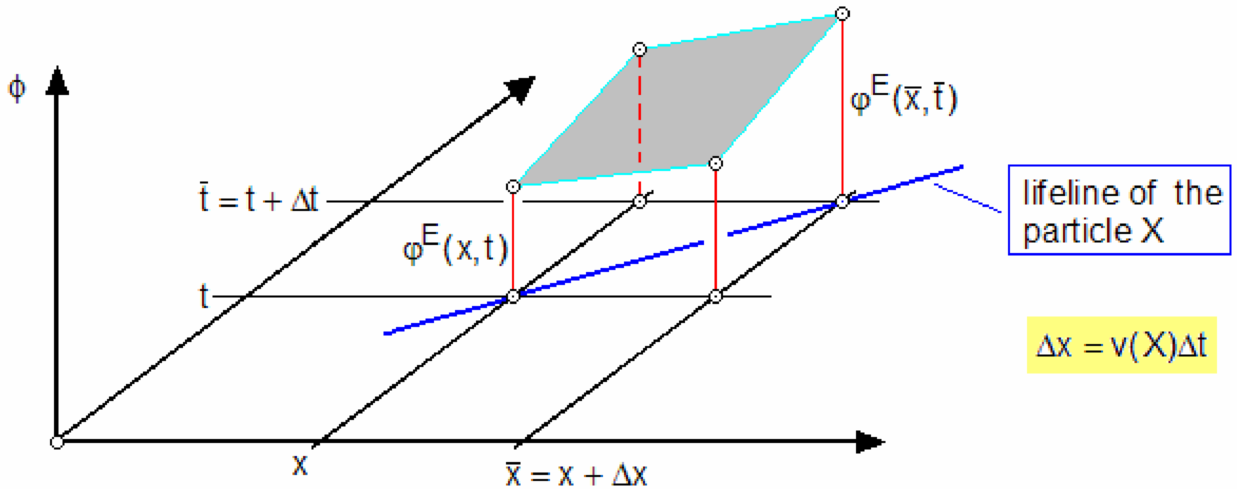
$$-\frac{\partial}{\partial x}(hv) = \frac{\partial h}{\partial t}$$

Momentum Balance



$$-\rho_w g h \frac{\partial h}{\partial x} = \rho_w h \dot{v}$$

the material time derivative $\dot{\phi} = D\phi / Dt$



$$\phi(x + v\Delta t, t + \Delta t) \approx \phi(x,t) + \left(\frac{\partial \phi}{\partial x}\right) v\Delta t + \left(\frac{\partial \phi}{\partial t}\right)\Delta t$$

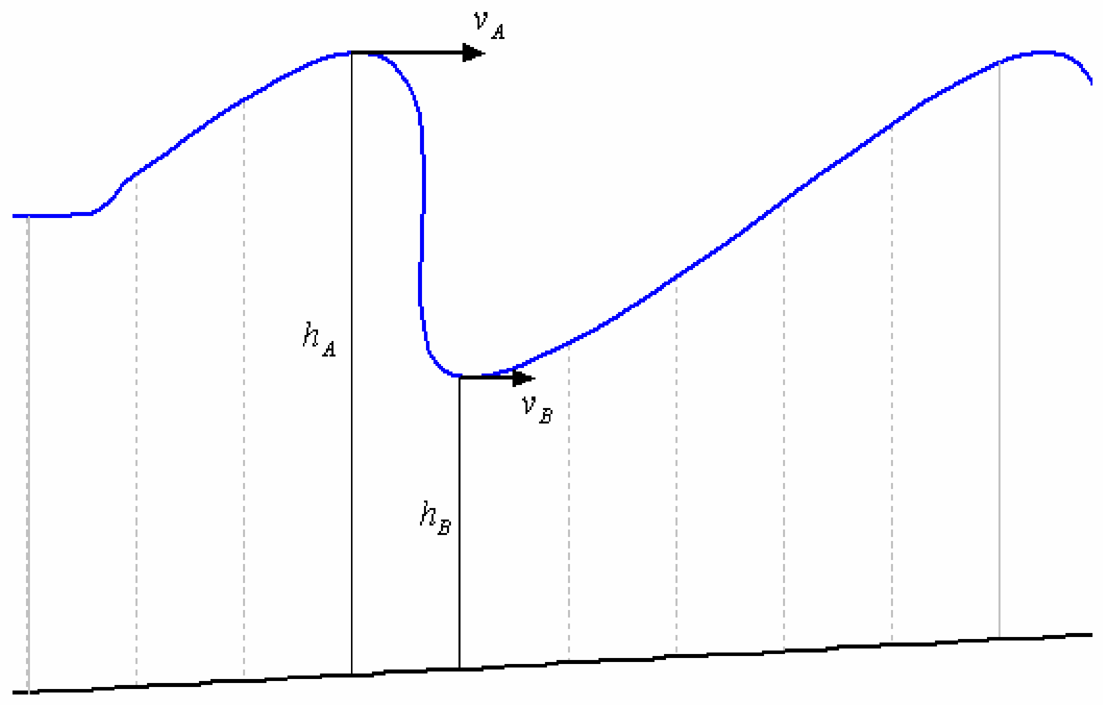
$$\frac{D\phi}{Dt} = \lim_{\Delta t \to 0}\left(\frac{\phi(x + v\Delta t, t + \Delta t) - \phi(x,t)}{\Delta t}\right) = \left(\frac{\partial \phi}{\partial x}\right) v + \left(\frac{\partial \phi}{\partial t}\right)$$

Kinematic water-waves

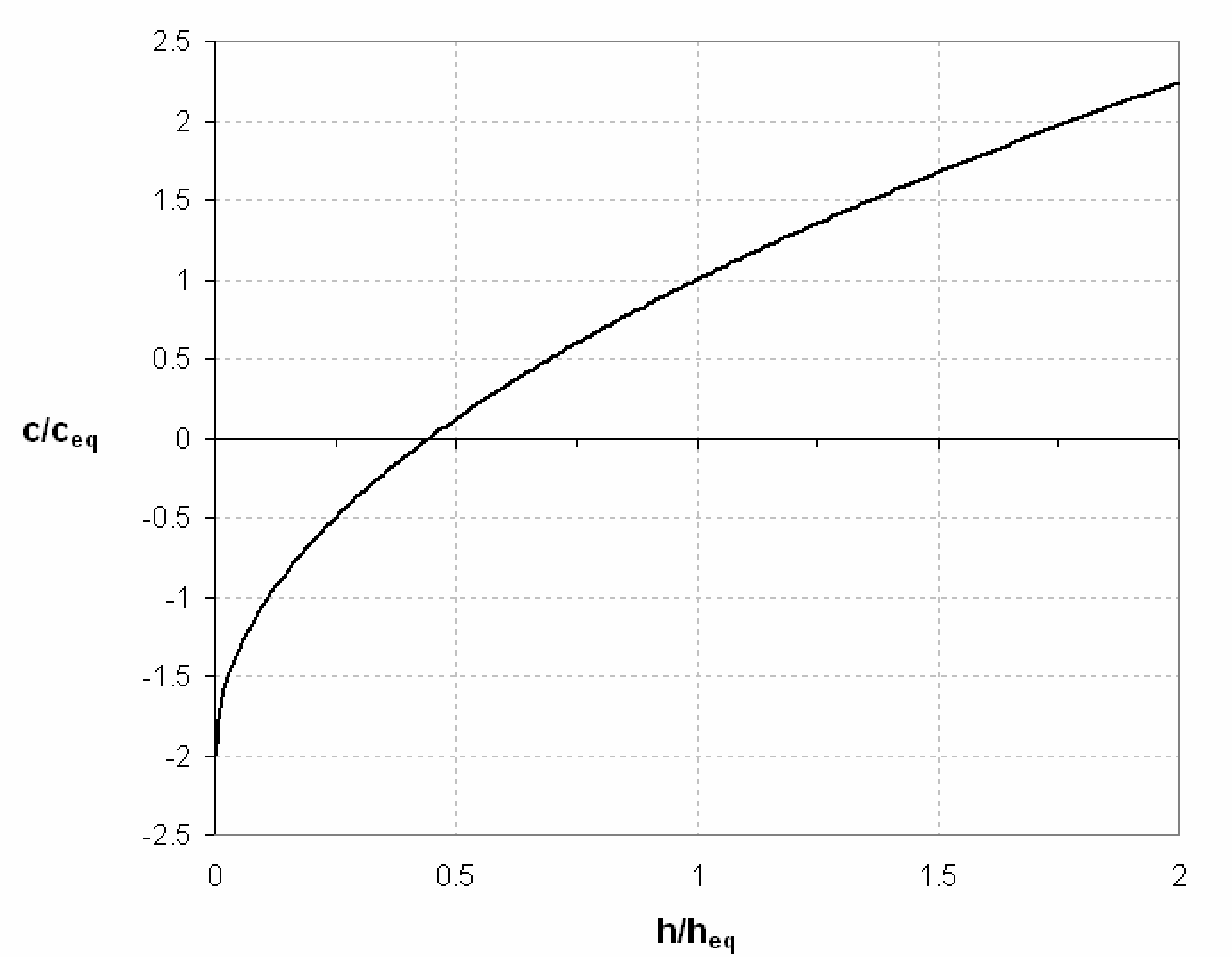
$$\frac{\partial h}{\partial t}+v\frac{\partial h}{\partial x}+h\frac{\partial v}{\partial x}=0$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial h}{\partial x}=0$$

$$v = V(h)$$



Kinematic water-waves



$$\frac{\partial h}{\partial t} + c(h)\frac{\partial h}{\partial x} = 0$$

$$c = 3c_{eq}\left(\sqrt{\frac{h}{h_{eq}}} - \frac{2}{3}\right)$$

$$c_{eq} = \sqrt{g h_{eq}}$$

Hyperbolic problems

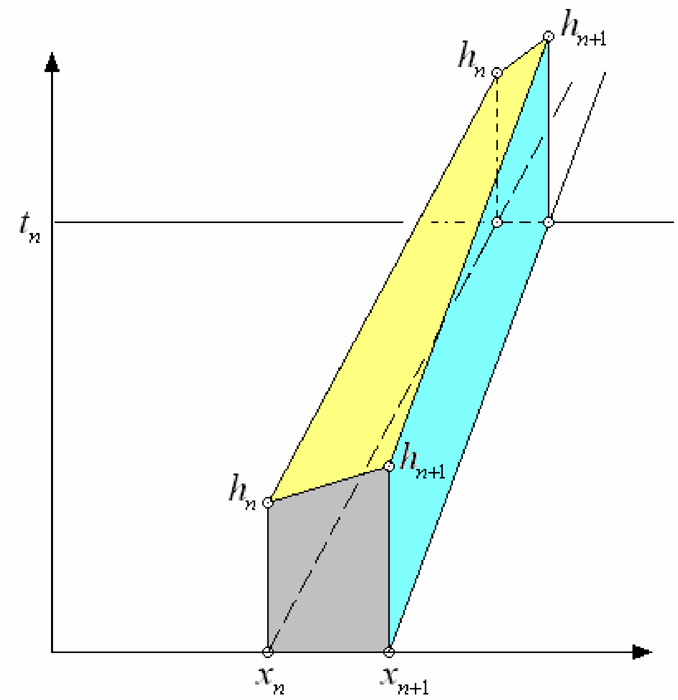
Numerical Procedure:

- Discretize the x-axis: $x_1 = a,\ x_2 = a + \Delta x, \ldots, x_{n+1} = a + n\Delta x, \ldots$
- Evaluate the initial condition for the height: $h_n = h_0(x_n), \quad n = 1, 2, \ldots$
- Compute the celerity: $c_n = 3\sqrt{g\,h_0(x_n)} - 2\sqrt{gh_0}$
- In the plane of events $O(x,t)$ and through the point on the x-axis, with co-ordinates (x_n, t) we draw the straight-line characteristic: $(\Gamma): x = x_n + c_n t$
- Along this characteristic straight line the information concerning the height and the velocity is transferred constant:

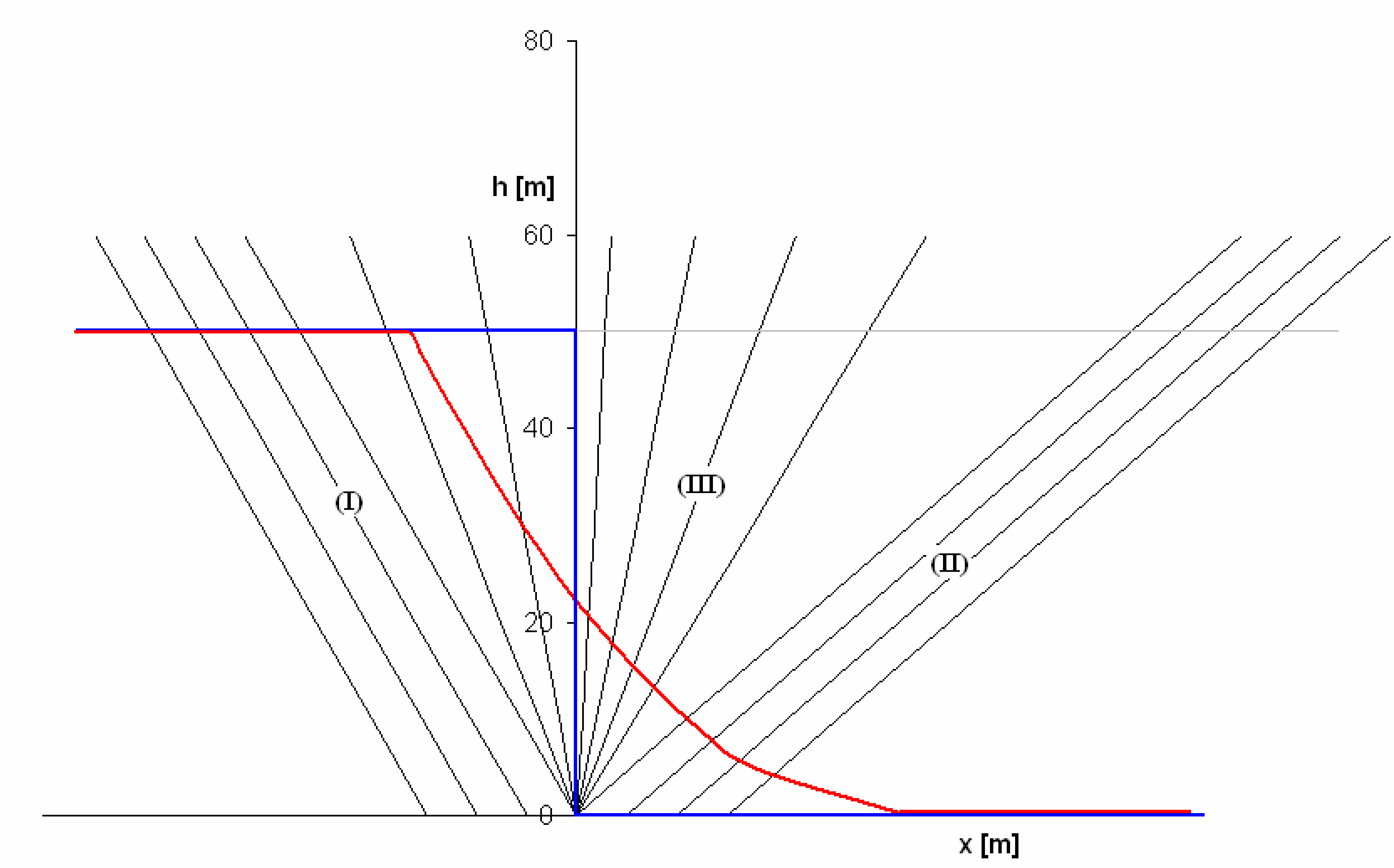
$$(\Gamma): h^E(x_n + c_n t, t) = h^E(x_n, 0) = H(x_n)$$

$$(\Gamma): v = V(x_n + c_n t, t) = 2\ \sqrt{g\,H(x_n)}\ -\ 2\ \sqrt{gh_0}$$

In order to evaluate the free-surface and velocity profiles at a given time $t = q > 0$, intersect the characteristics with the line, $t = q$, and plot over the intersection points the values for h and $v = V(h)$, which they carry.



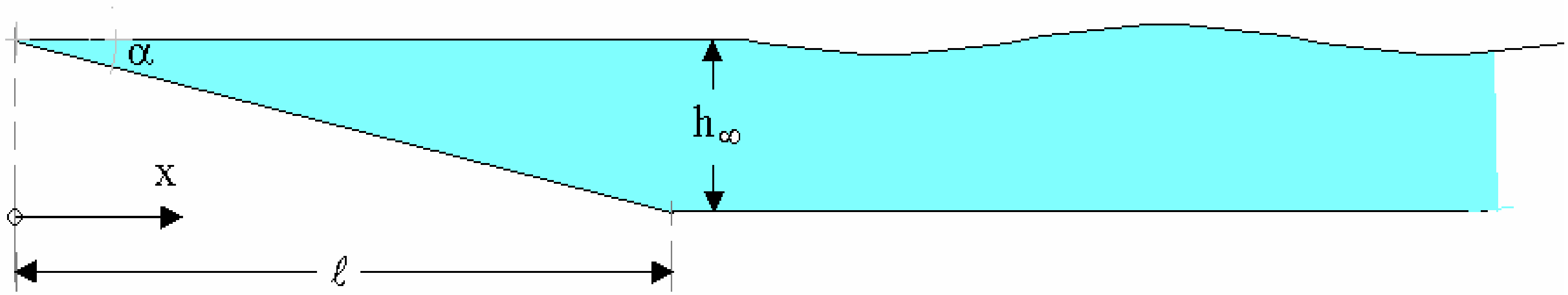
The “breaking-dam” problem



Tsunamis

(=harbor waves)

Long wave approaching a slopping beach (tsunami)



$$H = h_0 + \zeta(x,t)$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \approx \frac{\partial v}{\partial t}$$

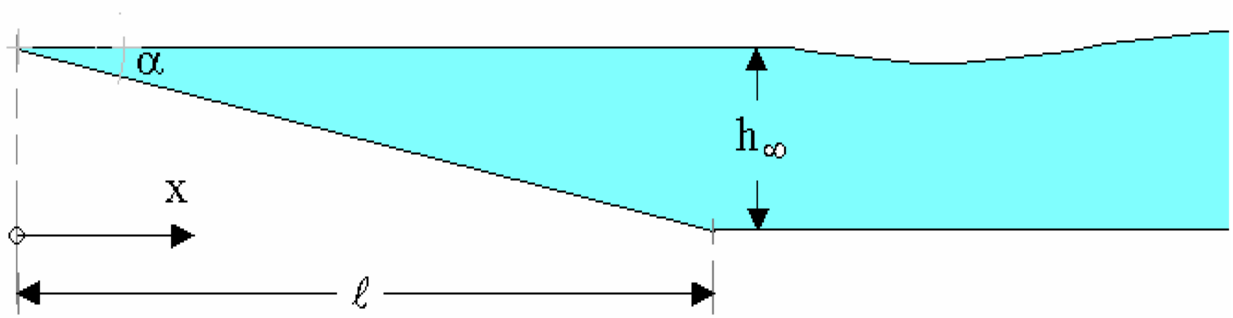
$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}\left((h_0 + \zeta)v\right) = 0 \quad \Rightarrow \quad \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(h_0 v) = 0$$

$$-g\frac{\partial H}{\partial x} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \quad \Rightarrow \quad \frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(h_0 v) = 0$$

$$\frac{\partial v}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0$$

$$h_0(x) = \begin{cases} \alpha x & (0 \leq x \leq \ell) \\ h_\infty & (\ell \leq x) \end{cases}$$



$$\frac{\partial^2\zeta}{\partial t^2} - g\alpha x\frac{\partial^2\zeta}{\partial x^2} - g\alpha\frac{\partial\zeta}{\partial x} = 0 \quad (0 \le x \le \ell)$$

$$\zeta = A(x)\cos(\omega t + \varepsilon) \ \ (0 \le x \le \ell) \qquad\qquad A(\ell) = a$$

$$x\frac{d^2A}{dx^2} + \frac{dA}{dx} + \frac{\omega^2}{\alpha g}A = 0$$

$$x = \frac{\alpha g}{\omega^2}\left(\frac{\xi}{2}\right)^2$$

$$\frac{d^2A}{d\xi^2} + \frac{1}{\xi}\frac{dA}{d\xi} + A = 0$$ **(Bessel differential equation)**

$$\frac{d^2A}{d\xi^2}+\frac{1}{\xi}\frac{dA}{d\xi}+A=0$$

Bowman, F. *Introduction to Bessel Functions*. Dover, 1958

$$A=C_1J_0(\xi)+C_2Y_0(\xi)$$

$J_0(\xi), Y_0(\xi)$ Bessel functions of 0^{th} order and 1^{st} and 2^{nd} kind respectively

$$J_0(\xi)=1-\frac{\frac{1}{4}\xi^2}{(1!)^2}+\frac{\left(\frac{1}{4}\xi^2\right)^2}{(2!)^2}-\frac{\left(\frac{1}{4}\xi^2\right)^3}{(3!)^2}+\cdots$$

$$Y_0(\xi)=\frac{2}{\pi}\left(\ln\left(\frac{\xi}{2}\right)+\gamma\right)J_0(\xi)+\frac{2}{\pi}\left\{\frac{\left(\frac{1}{4}\xi^2\right)}{(1!)^2}-\left(1+\frac{1}{2}\right)\frac{\left(\frac{1}{4}\xi^2\right)^2}{(2!)^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\left(\frac{1}{4}\xi^2\right)^3}{(3!)^2}-\cdots\right\}$$

$\gamma=0.5772156649\ldots$ Euler constant

For a bounded solution at $\xi = 0$: $|A(0) < \infty|$ the logarithmic or Neumann-Weber solution $Y_0(\xi)$ is excluded

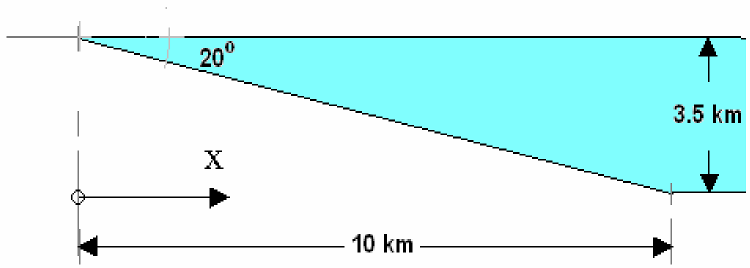
$$C_2 = 0$$

$$A = C_1 J_0(\xi)\ ,\ \xi = 2\omega\sqrt{\frac{x}{\alpha g}}$$

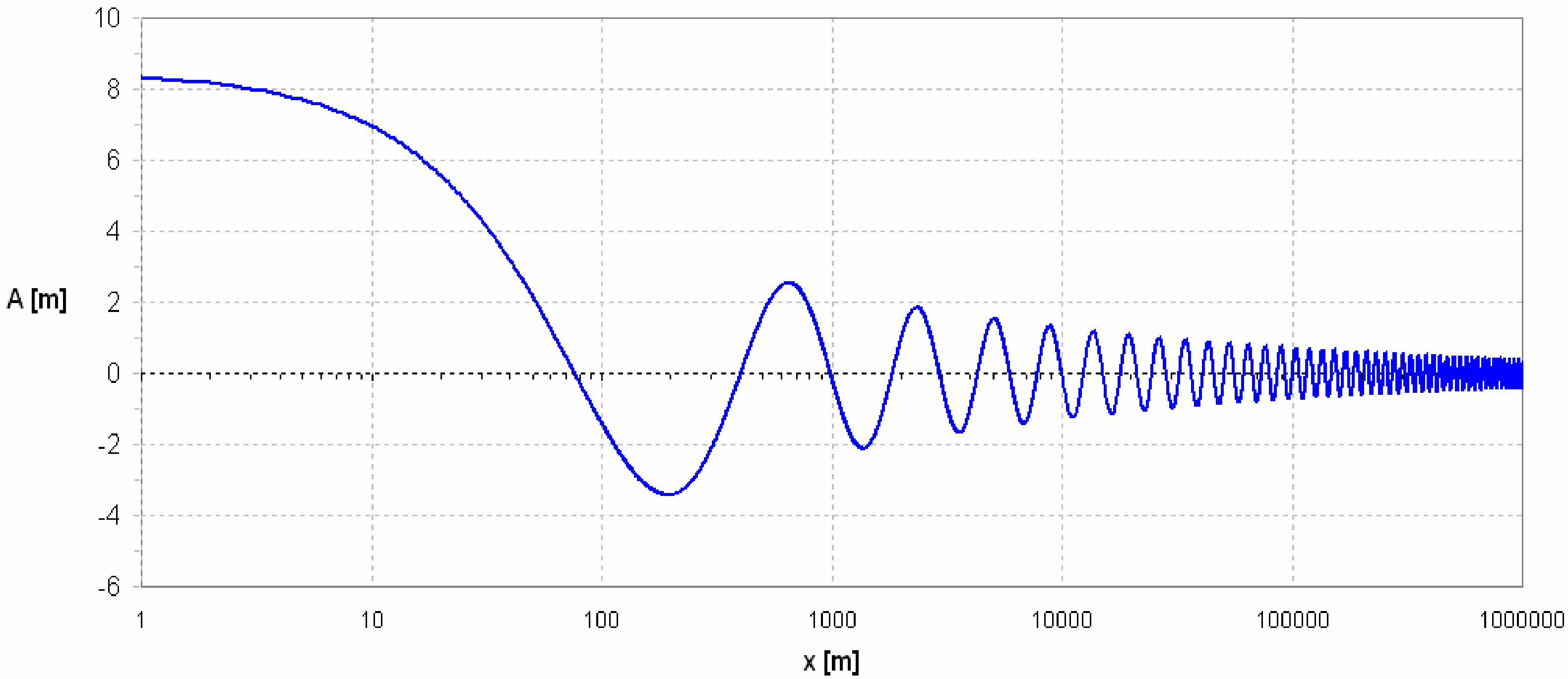
$$\Rightarrow \quad \zeta = a\frac{J_0\left(2\omega\sqrt{\frac{x}{\alpha g}}\right)}{J_0\left(2\omega\sqrt{\frac{\ell}{\alpha g}}\right)}\cos(\omega t + \varepsilon)$$

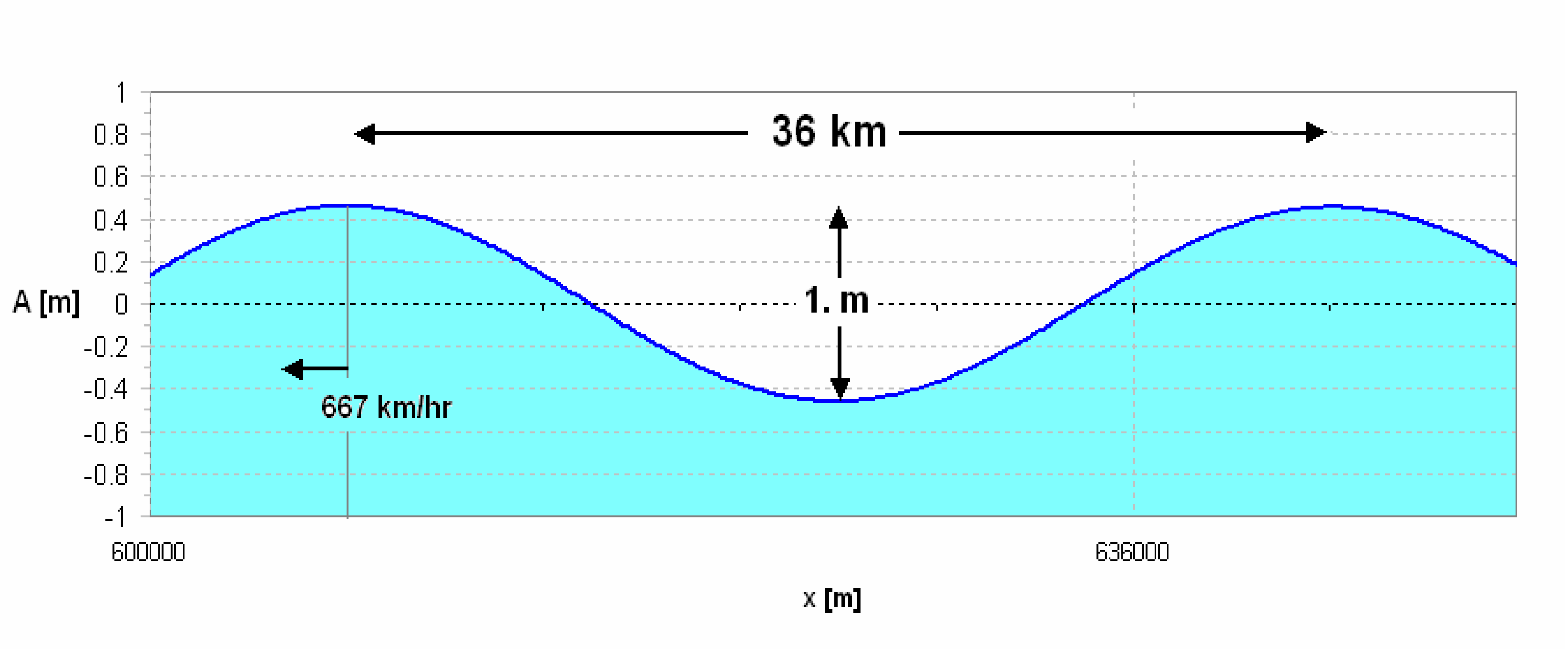
the tsunami

$$c = \sqrt{g h_0}$$



l	10000	[m]		
hinf	3500	[m]		
α	19.29	[deg]		
ω	0.25	[1/s]		
T	25.13274	s	0.42	min
c	185.2971	m/s	667.07	km/hr





36 km
1
0.8
0.6
0.4
0.2
A [m]
0
-0.2
-0.4
-0.6
-0.8
-1
1. m
667 km/hr
600000
636000
x [m]

$$\Rightarrow \quad \zeta = a\frac{J_0\left(2\omega\sqrt{\frac{x}{\alpha g}}\right)}{J_0\left(2\omega\sqrt{\frac{\ell}{\alpha g}}\right)}\cos(\omega t+\varepsilon)$$

roots of the Bessel function J_0

j01	2.404825558
j02	5.52007811
j03	8.653727913
j04	11.79153444
j05	14.93091771
j06	18.07106397
j07	21.21163663
j08	24.35247913
j09	27.49347913
j10	30.63460647
j11	33.77582021
j12	36.91709835
j13	40.05842576
j14	43.19979171
j15	46.34118837

$$\Omega = 2\omega\sqrt{\frac{\ell}{\alpha g}} = j_0^{(i)}$$ **resonance condition**

in our example we $\Omega = 26.98$

Question

Given that for a traffic-flow problem the velocity-density relationship is linear determine the traffic flow density so that the car flow is optimal